1. (**2.5 pts. each**) Which of the following equation define operations on the set of integers? Of those that do, which are associative? Which are commutative? Which have identity elements?

(a) $a * b = 3$.

(b) $a * b = a$.

Solution:

(a)
- For any $a, b \in \mathbb{Z}$, we have $a * b = 3 \in \mathbb{Z}$. So, $*$ is an operation.
- For any $a, b, c \in \mathbb{Z}$, we have $(a*b)*c = 3*c = 3$. On the other hand, $a*(b*c) = a*3 = 3$. Therefore, $*$ is associative.
- For any $a, b \in \mathbb{Z}$, we have $a * b = 3 = b * a$. So, $*$ is commutative.
- If $a = 1$, then $a * e = 3 \neq 1$. Thus, $*$ has no identity.

(b) $a * b = a$.
- For any $a, b \in \mathbb{Z}$, we have $a * b = a \in \mathbb{Z}$. So, $*$ is an operation.
- For any $a, b, c \in \mathbb{Z}$, we have $(a*b)*c = a*c = a$. On the other hand, $a*(b*c) = a*b = a$. Therefore, $*$ is associative.
- For any $a, b \in \mathbb{Z}$, we have $a * b = a$ where $b * a = b$. So, $*$ is not commutative.
- If $a \neq e$, then $e * a = e \neq a$. Thus, $*$ has no identity.

2. (**1+2+2 pts.**) In S_6:

(a) Decide whether $\alpha = (1\ \ 3\ \ 5)\,(2\ \ 4\ \ 6)$ is an even or an odd permutation.

(b) Find the cyclic decomposition of $\beta = (1\ \ 3\ \ 5)\,(2\ \ 3\ \ 4\ \ 6)$.

(c) Compute $\gamma = (1\ \ 2\ \ 3)^{-1}\,(4\ \ 5\ \ 6)\,(1\ \ 3\ \ 2)$.

Solution:

(a) Simply $\alpha = (1\ \ 5)(1\ \ 3)(2\ \ 6)(2\ \ 4)$. Thus the permutation is even.

(b) Do a direct multiplication to get: $\beta = (1\ \ 3\ \ 4\ \ 6\ \ 2\ \ 5)$ which is a one cycle.

(c) $\gamma = (1\ \ 3\ \ 2)(4\ \ 5\ \ 6)(1\ \ 3\ \ 2) = (1\ \ 2\ \ 3)(4\ \ 5\ \ 6)$.

3. (**5 pts.**) Let $S = \{1, 2, 3, 4, 5, 6, 7, 8\}$, $G = Sym(S)$, and $T = \{1, 3, 5\} \subseteq S$. Find

$$G_{(T)} = \{ \alpha \in G : \alpha(T) = T \}$$

which leaves T setwise invariant.

Solution: Note that the order of $G_{(T)}$ is $3!\,5! = 720$ which means that it is very difficult to list all the elements. So, simply let $A = T$ and let $B = S - T$. Then

$$G_{(T)} = \{ \alpha\beta \, : \, \alpha \in Sym(A) \text{ and } \beta \in Sym(B) \}.$$

4. (3+2 pts.) Let M and N be two subgroups of a group G.

(a) Show that $M \cap N$ is a subgroup of G.

(b) Is $M \cap N$ a subgroup of M? Explain your answer.

Solution:

(a) We proof the three conditions as follows:

($\mathcal{S}_1$) M and N are subgroups and both contains the identity element. Thus $e \in M \cap N \neq \phi$.

($\mathcal{S}_2$) Let $a, b \in M \cap N$. Then $a, b \in M$ $(a, b \in N)$ and hence $ab \in M$ $(ab \in N)$ since both M and N are subgroups. Therefore, $ab \in M \cap N$.

($\mathcal{S}_3$) Let $a \in M \cap N$. Then $a \in M$ $(a \in N)$. Hence $a^{-1} \in M$ $(a^{-1} \in N)$. Thus, $a^{-1} \in M \cap N$.

Therefore, $M \cap N$ is a subgroup of G.

(b) Yes. Since $M \cap N$ is a group and $M \cap N \subseteq M$. That is $M \cap N \leq M$.

5. (**3 pts. each**) Let G be a group.

(a) If $(ab)^2 = a^2b^2$, for all $a, b \in G$, then show that G must be abelian.

(b) If $a = a^{-1}$, for all $a \in G$, then show that G must be abelian.

Solution:

(a) Note that $(ab)^2 = abab = a^2b^2$. Multiplying both sides by a^{-1} from left and b^{-1} from right, we get: $ba = ab$. That is G is abelian.

(b) Let $a, b \in G$. Then $(ab)^{-1} = ab$, $a^{-1} = a$, and $b^{-1} = b$. Therefore,

$$ab = (ab)^{-1} = b^{-1}a^{-1} = ba.$$