Kuwait University
Faculty of Science
Department of Mathematics

Euclidean Geometry 0410-226
Final Exam

Monday, December 16, 2019
Fall 2019/2020

Name										
ID Number										
Serial Number										

Duration **2** hours (This exam contains **6** questions).

Section No.	Instructor Name
1	**Dr. Abdullah Alazemi**

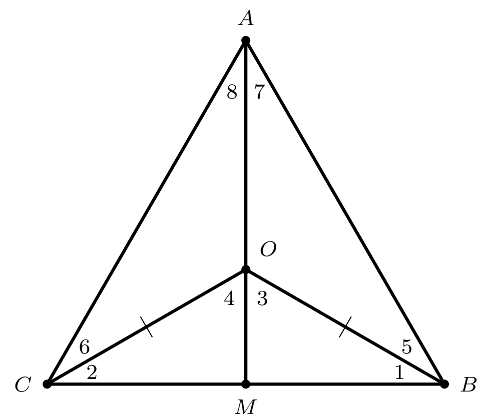
Calculators and communication devices are not allowed in the examination room.
Give full reasons for your answer and State clearly any Theorem you use.

Question 1	/6
Question 2	/9
Question 3	/6
Question 4	/10
Question 5	/9
Question 6	/2
Total	/42

Good Luck

1. (**6 pts.**) In the diagram, let $\overline{AM}$ bisect the angle $C\hat{O}B$. Assume that $\triangle\, BCO$ is an isosceles triangle. Show that $\triangle\, ABM \cong \triangle\, ACM$.

Solution:



In triangles $\triangle\, OBM$ and $\triangle\, OCM$, we have:

(a) $\hat{3} \cong \hat{4}$ ($\overline{AM}$ is a bisector).

(b) $\overline{OB} \cong \overline{OC}$ (given - $\triangle\, BCO$ is isocseles).

(c) $\hat{1} \cong \hat{2}$ (given - $\triangle\, BCO$ is isocseles).

By ASA, we have $\triangle\, OBM \cong \triangle\, OCM$. That is $\overline{BM} \cong \overline{CM}$ and $B\hat{M}A \cong C\hat{M}A$ (both right angles). Now, in triangles $\triangle\, ABM$ and $\triangle\, ACM$, we have:

(a) $\overline{AM}$ - common.

(b) $A\hat{M}B \cong A\hat{M}C$ (proved).

(c) $\overline{MB} \cong \overline{MC}$ (proved).

By SAS, we have $\triangle\, ABM \cong \triangle\, ACM$.

2. (**3 pts. each**)

(a) Let $\overline{RS}$ be tangent to $\odot A$ and $\odot B$. Show that $\triangle\, ARC \sim \triangle\, BSC$.

(b) If $\left|\,\overline{BS}\,\right| = 0.5\left|\,\overline{AR}\,\right|$, define a transformation that maps $\odot B$ to $\odot A$.

(c) Let $\overline{AB}$ be a diameter of the circle $\odot O$, $\left|\,\widehat{\mathrm{BC}}\,\right| = 40°$, and $\left|\,\hat{1}\,\right| = 60°$. Find $\left|\,\widehat{\mathrm{BD}}\,\right|$.

<u>Solution</u>:

Figure: Part (a) and (b)

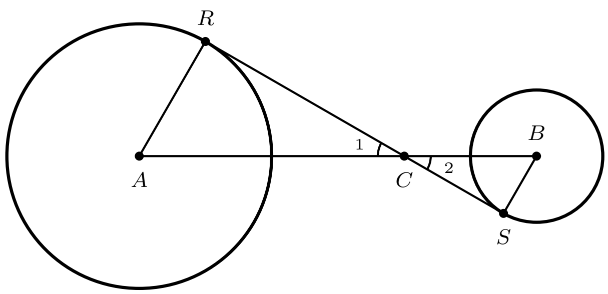
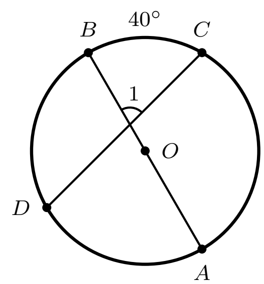


Figure: Part (c)



(a) Clearly, $\hat{1} \cong \hat{2}$ (vertically opposite). Also $\hat{S} \cong \hat{R}$ (both are right angles). By S-AA, $\triangle\, ARC \sim \triangle\, BSC$.

(b) A dilation $\mathcal{D}_{C,-2}$ would maps $\odot B$ to $\odot A$.

(b) Clearly, by Theorem in class $\left|\,\hat{1}\,\right| = \frac{1}{2}\left(\left|\,\widehat{\mathrm{BC}}\,\right| + \left|\,\widehat{\mathrm{AD}}\,\right|\right)$. That is $60° = \frac{1}{2}(40° + \left|\,\widehat{\mathrm{AD}}\,\right|)$. Hence $\left|\,\widehat{\mathrm{AD}}\,\right| = 120° - 40° = 80°$. Therefore, $\left|\,\widehat{\mathrm{BD}}\,\right| = 100°$

3. (**3 pts. each**)

(a) Let O, P and Q be three noncollinear points so that $\mathcal{R}_{O,\theta}(O) = O$, $\mathcal{R}_{O,\theta}(P) = P'$ and $\mathcal{R}_{O,\theta}(Q) = Q'$. Show that $\overline{PQ} \cong \overline{P'Q'}$.

(b) Assume that $\triangle ABC$ and $\triangle DEC$ are two isosceles right triangles. Show that $\overline{AD} \cong \overline{BE}$.

Solution:

Figure: Part (a)

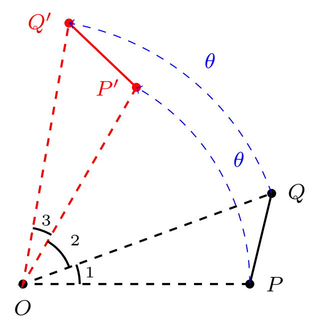
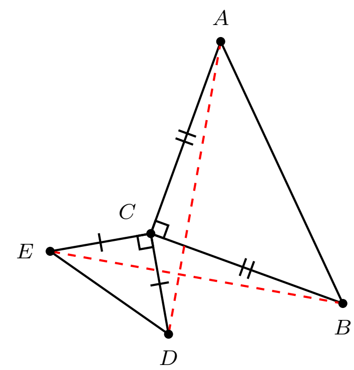


Figure: Part (b)



(a) We need to show that $\left|\overline{PQ}\right| = \left|\overline{P'Q'}\right|$. In $\triangle POQ$ and $\triangle P'OQ'$, we have:

i. $\left|\overline{OP}\right| = \left|\overline{OP'}\right|$ and $\left|\overline{OQ}\right| = \left|\overline{OQ'}\right|$ (definition of rotation).

ii. $\left|\hat{1}\right| = \theta - \left|\hat{2}\right| = \left|\hat{3}\right|$ (look at diagram).

By SAS, $\triangle POQ \cong \triangle P'OQ'$. That is $\left|\overline{PQ}\right| = \left|\overline{P'Q'}\right|$.

(b) Note that $\mathcal{R}_{C,90}(B) = A$ and $\mathcal{R}_{C,90}(E) = D$. Hence, under the same rotation, we have $\overline{BE}$ maps to $\overline{AD}$. That is $\mathcal{R}_{C,90}\left(\overline{BE}\right) = \overline{AD}$. Therefore, $\overline{AD} \cong \overline{BE}$ since rotation is an isometry (or by part a).

4. (4+3+3 pts.)

(a) Let $l_1 : \; x + y - 1 = 0$ and $l_2 : \; x - y - 1 = 0$ be two lines. Find all points that are equidistants from l_1 and l_2.

(b) Use lines l_1 and l_2 (from part a) to write the product $\mathbf{R}_{l_1} \circ \mathbf{R}_{l_2}$ as a single rotation centered at the intersection point (if any) of the two lines.

(c) Find an equation of the circle with center $A(-1, 2)$ and tangent to the y-axis. Furthermore, **if any** find the point(s) of intersection of the circle and x-axis.

<u>**Solution**</u>:

(a) Let $M(x, y)$ be the points of the locus. Thus,

$$d(M, l_1) = d(M, l_2) \quad \Rightarrow \quad \frac{|\, x + y - 1\,|}{\sqrt{1+1}} = \frac{|\, x - y - 1\,|}{\sqrt{1+1}} \quad \Rightarrow |\, x + y - 1\,| = |\, x - y - 1\,|\,.$$

That is we have two cases:

Case 1: $(\, x + y - 1\,) = +(\, x - y - 1\,)$, and hence $y = 0$.

Case 2: $(\, x + y - 1\,) = -(\, x - y - 1\,)$, and hence $x = 1$.

Therefore, all points that are equidistant from l_1 and l_2 lie on the two lines $y = 0$ and $x = 1$

(b) Note that the slopes of l_1 and l_2 are -1 and 1, respectively. Thus, the two lines are perpendicular. Thus the angle between the two lines is $90°$. Moreover, the two lines are intersecting in the point $(1, 0)$. Therefore, $\mathbf{R}_{l_1} \circ \mathbf{R}_{l_2} = \mathcal{R}_{(1,0),180°}$.

(c) Since the circle is tangent to y-axis, we have the radius equals to the x-coordinates of A which is the distance from y-axis to A. Thus, $r = |-1| = 1$, and hence the circle equation: $(x + 1)^2 + (y - 2)^2 = 1$.

Substitute $y = 0$ in the circle equation to get $(x + 1)^2 = -4$. This is not possible. Hence, there are no intersection points.

5. (**3 pts. each**) Find the image of the circle $(x-1)^2+(y-3)^2=1$ under each of the following:

(a) A reflection in the line $y-3x=0$.

(b) A rotation $\mathcal{R}_{(1,-1),\frac{\pi}{2}}$.

(c) A dilation $\mathcal{D}_{O,-3}$.

<u>**Solution**</u>:

(a) Note that the center of the circle $P(1,3)$ lies on the line of reflection. Therefore, the circle is invariant, and hence the image is again: $(x-1)^2+(y-3)^2=1$.

(b) The rotation matrix about the origin through the angle $\frac{\pi}{2}$ is given by $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. But we are rotating about $(1,-1)$. Thus the image of $P(1,3)$ is $P'(x,y)$ where

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \left(\begin{bmatrix} 1 \\ 3 \end{bmatrix} - \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right) + \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -3 \\ -1 \end{bmatrix}$$

Therefore, the image of the circle is: $(x+3)^2+(y+1)^2=1$.

(c) Dilation is **not** an isometry. Hence the new radius $r' = |\lambda|1 = |-3| = 3$.

Also, $\mathcal{D}_{O,-3}(A(1,3)) = A'(-3,-9)$. Thus,

$$c'(A',r'):\ (x+3)^2+(y+9)^2=9.$$

6. (**2 pts.**) **Bonus Question**: Give a detailed definition of a homothecy of the Euclidean Plane.

Solution: Let λ be a nonzero scalar. A **homothecy** (or **homothety**, or **dilation**), denoted $\mathcal{D}_{O,\lambda}$, is the transformation that maps O to itself and for any other point P,

$$P \mapsto \begin{cases} P' \in \overrightarrow{OP}, & \text{if } \lambda > 0; \\ P' \in \overrightarrow{PO}, & \text{if } \lambda < 0; \end{cases}$$

such that $\left|\overline{OP'}\right| = |\,\lambda\,|\ \left|\overline{OP}\right|$. The point O and the scalar λ are called the center of and the ratio of the homothecy, respectively.