1. (**3 pts. each**) Let $\triangle\, ABC$ be a triangle.

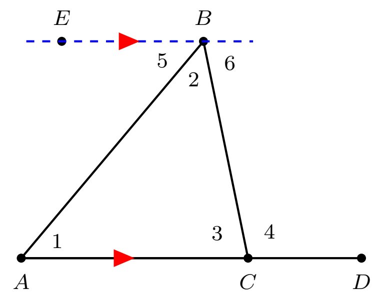
(a) Show that the measure of angles of $\triangle\, ABC$ sums to $180°$.

(b) Show that $|\,\hat{4}\,| = |\,\hat{1}\,| + |\,\hat{2}\,|$ (drawn).

Solution:

(a) Draw a line $\overleftrightarrow{BE}$ parallel to $\overleftrightarrow{AC}$, see the diagram. Note that $|\,\hat{2}\,| + |\,\hat{5}\,| + |\,\hat{6}\,| = 180°$ (supp. angles). The line $\overleftrightarrow{AB}$ is a transversal to the parallel lines $\overleftrightarrow{BE}$ and $\overleftrightarrow{AC}$. Hence, $\hat{1} \cong \hat{5}$ (alternate interior angles). Also, $\overleftrightarrow{BC}$ is another transversal and hence $\hat{3} \cong \hat{6}$. Thus, $|\,\hat{2}\,| + |\,\hat{1}\,| + |\,\hat{3}\,| = |\,\hat{2}\,| + |\,\hat{5}\,| + |\,\hat{6}\,| = 180°$.

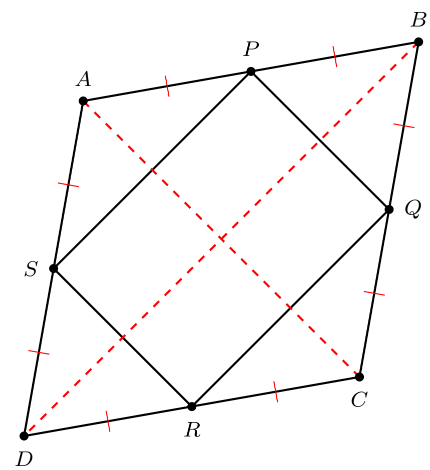
(b) Note that $\hat{3}$ and $\hat{4}$ are supplementary angles and hence $|\,\hat{3}\,| + |\,\hat{4}\,| = 180°$. Thus, $|\,\hat{1}\,| + |\,\hat{2}\,| + |\,\hat{3}\,| = 180° = |\,\hat{3}\,| + |\,\hat{4}\,|$. Therefore, $|\,\hat{4}\,| = |\,\hat{1}\,| + |\,\hat{2}\,|$.



2. (**3 pts. each**) If $ABCD$ is a rhombus with midpoints P, Q, R, and S of the sides.

(a) Show that $PQRS$ is a parallelogram.

(b) Show that $PQRS$ is a rectangle.



Solution:

(a) Note that in $\triangle ABC$, we have P and Q are the midpoints of $\overline{AB}$ and $\overline{BC}$, respectively. Hence, $\overline{PQ} \parallel \overline{AC}$ (Theorem in class). Moreover,

i. $\dfrac{|\overline{BP}|}{|\overline{BA}|} = \dfrac{|\overline{BQ}|}{|\overline{BC}|} = \frac{1}{2}$.

ii. $\hat{B}$ is common.

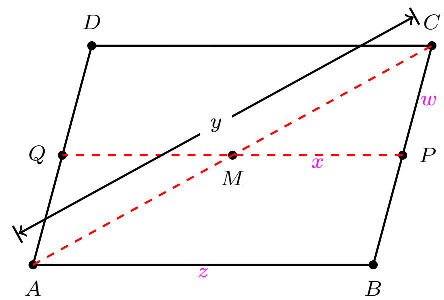
Thus we get $\triangle BPQ \sim \triangle BAC$ by S-SAS. That is $\dfrac{|\overline{PQ}|}{|\overline{AC}|} = \frac{1}{2}$. Do the same procedure on $\triangle DAC$ and $\triangle DSR$, we get: $\overline{SR} \parallel \overline{AC}$ and $\dfrac{|\overline{SR}|}{|\overline{AC}|} = \frac{1}{2}$. Therefore, $\overline{PQ} \parallel \overline{RS}$ and $\overline{PQ} \cong \overline{RS}$. Thus, $PQRS$ is a parallelogram.

(b) From part (a), we proved that $\overline{PQ} \parallel \overline{AC}$. We can (in the same way as in part (a)) show that $\overline{PS} \parallel \overline{BD}$. But $\overline{AC} \perp \overline{BD}$ (since $ABCD$ is a rhombus). Therefore, $\overline{PQ} \perp \overline{PS}$ and hence $Q\hat{P}S$ is a right angle. Since $\overline{PS} \parallel \overline{QR}$, we also get $P\hat{Q}R$ is right angle as well. Therefore, $PQRS$ is a rectangle.

3. (**3 pts. each**) Let $ABCD$ be a parallelogram with Q in $\overline{AD}$ and P in $\overline{BC}$ so that $\overline{PQ} \parallel \overline{AB}$.

(a) Find $|\,MC\,|$ in terms of x, y, z, w.

(b) Show that $|\,\overline{AM}\,| \cdot |\,\overline{CP}\,| = |\,\overline{AQ}\,| \cdot |\,\overline{CM}\,|$.



Solution:

(a) Triangle Proportionality Theorem implies $\triangle\, ABC \sim \triangle\, MPC$.

That is
$$\frac{|\,\overline{AB}\,|}{|\,\overline{MP}\,|} = \frac{|\,\overline{AC}\,|}{|\,\overline{MC}\,|} = \frac{|\,\overline{BC}\,|}{|\,\overline{PC}\,|}.$$

Hence, $\dfrac{z}{x} = \dfrac{y}{|\,\overline{MC}\,|} = \dfrac{|\,\overline{BC}\,|}{w}$. Therefore, $|\,\overline{MC}\,| = \dfrac{xy}{z}$.

(b) Note that $\triangle\, AMQ$ and $\triangle\, CMP$ have:

- $A\hat{M}Q \cong P\hat{M}C$ (vertically opposite angles).
- $A\hat{Q}M \cong C\hat{P}M$ (alternate interior angle since $\overline{AD} \parallel \overline{BC}$).

Therefore, $\triangle\, AMQ \sim \triangle\, CMP$. Thus, $\dfrac{|\,\overline{AM}\,|}{|\,\overline{CM}\,|} = \dfrac{|\,\overline{AQ}\,|}{|\,\overline{CP}\,|}$ and hence the result.

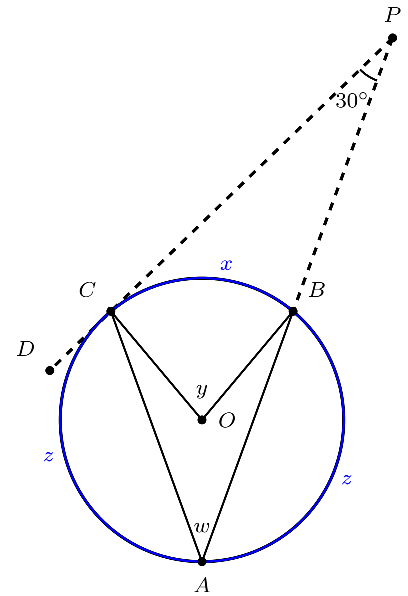
4. (**3+3+2 pts.**) In the diagram, let $\left|\hat{P}\right| = 30°$. Let $\overline{PD}$ be a tangent line to the circle $\odot O$ at the point C. Also assume that $\overset{\frown}{\mathrm{AB}} \cong \overset{\frown}{\mathrm{AC}}$.

(a) Find x, y, z and w.

(b) Show that $\triangle\, ABO \cong \triangle\, ACO$.

(c) Find $\left|A\hat{C}O\right|$.

<u>**Solution**</u>:



(a) Note that $\left|\hat{P}\right| = 30° = \frac{1}{2}(z - x)$ (Theorem in the class) which implies that $60° = z - x$ or $z = 60° + x$. Moreover, $x + 2z = 360°$ implies that $x + 2(60° + x) = 360$. That is, $3x = 240°$. Therefore, $x = 80°$ and $z = 140°$. Therefore, $y = x = 80°$ (central angle), and $w = \frac{1}{2}x = 40°$ (inscribed angle).

(b) in the triangles $\triangle\, ABO$ and $\triangle\, ACO$, we have:

i. $\overline{BO} \cong \overline{CO}$ (both are radii).

ii. $\overline{AB} \cong \overline{AC}$ (since $\overset{\frown}{\mathrm{AB}} \cong \overset{\frown}{\mathrm{AC}}$).

iii. $\overline{AO}$ is common.

By SSS, $\triangle\, ABO \cong \triangle\, ACO$.

(c) Clearly $\left|A\hat{C}D\right| = \frac{1}{2}z = 70°$ (Theorem in the class). But $\left|O\hat{C}D\right| = 90° = \left|A\hat{C}O\right| + \left|A\hat{C}D\right|$. Hence $\left|A\hat{C}O\right| = 20°$.