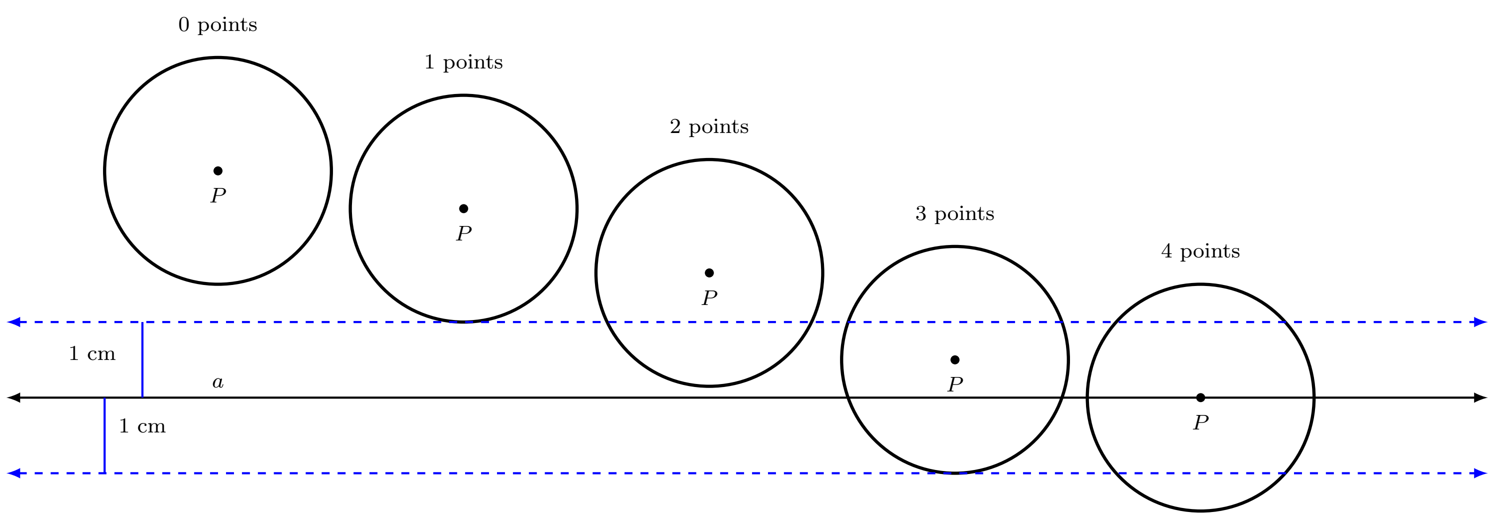
1. (**4 pts.**) Given a point P and a line a, what is the locus of points 3 cm from P and 1 cm from a? Justify your answer.

Solution:

The locus is 0, 1, 2, 3, or 4 points, depending on the intersection of $\odot P$ with radius 3 cm, and two paralell lines to a at distance 1 cm.



2. (**2 pts. each**) Let $\mathbf{T}$ and $\mathbf{S}$ be two isometries of the plane.

(a) Show that $\mathbf{S} \circ \mathbf{T}$ is an isometry.

(b) Show that $\mathbf{T}^{-1}$ is an isometry.

(c) Show that if $\mathbf{T}$ and $\mathbf{S}$ agrees on three noncollinear points, then they are identical.

<u>**Solution**</u>:

(a) For any A and B, $\mathbf{T}\left(\overline{AB}\right) = \overline{A'B'}$ and $\mathbf{S}\left(\overline{A'B'}\right) = \overline{A''B''}$, where $\overline{AB} \cong \overline{A'B'}$ ($\mathbf{T}$ is isometry) and $\overline{A'B'} \cong \overline{A''B''}$ ($\mathbf{S}$ is isometry).
Therefore, $\mathbf{ST}\left(\overline{AB}\right) = \mathbf{S}\left(\mathbf{T}\left(\overline{AB}\right)\right) = \mathbf{S}\left(\overline{A'B'}\right) = \overline{A''B''}$, with $\overline{AB} \cong \overline{A''B''}$.

(b) Assume that $\mathbf{T}\left(\overline{AB}\right) = \overline{A'B'}$ with $\overline{AB} \cong \overline{A'B'}$. Then clearly, $\mathbf{T}^{-1}\left(\overline{A'B'}\right) = \overline{AB}$ and $\overline{A'B'} \cong \overline{AB}$.

(c) Assume that $\mathbf{T}(A) = \mathbf{S}(A)$, $\mathbf{T}(B) = \mathbf{S}(B)$, $\mathbf{T}(C) = \mathbf{S}(C)$, for noncollinear points A, B, C. Then $\mathbf{S}^{-1}\mathbf{T}(A) = A$, $\mathbf{S}^{-1}\mathbf{T}(B) = B$, $\mathbf{S}^{-1}\mathbf{T}(C) = C$. That is $\mathbf{S}^{-1}\mathbf{T} = \mathbf{I}$. Hence, $\mathbf{T} = \mathbf{S}$.

3. (**4 pts. each**) Let $\mathbf{R}_l$ denote a reflection in a line l.

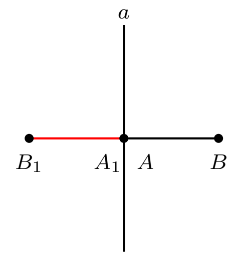
(a) Let A and B be points in the plane so that $\mathbf{R}_a(A) = A_1 = A$, and $\mathbf{R}_a(B) = B_1 \neq B$. Use the definition of reflection to show that $\overline{AB} \cong \overline{A_1B_1}$.

(b) Let a, b, r be three lines in the plane. If $\mathbf{R}_r(a) = b$, show that $a \parallel r$ iff $b \parallel r$.

<u>**Solution**</u>:

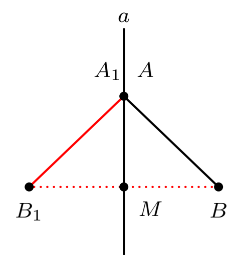
(a) Note that $A \in a$ since $\mathbf{R}_a(A) = A_1 = A$. Then, we have two cases for B as follows:

- $a \perp \overline{AB}$: $B \notin a$ and hence line a is perpendicular bisector of $\overline{BB_1}$ (by the definition of reflection). Therefore, A is the midpoint of $\overline{BB_1}$ and hence $|\overline{AB}| = |\overline{A_1B_1}|$.



- $a \not\perp \overline{AB}$: a is the perpendicular bisector of $\overline{BB_1}$. Let $M \in a$ be the midpoint of $\overline{BB_1}$. In right triangles $\triangle\, ABM$ and $\triangle\, A_1B_1M$, we have
  i. $\overline{AM}$ is common.
  ii. $\overline{BM} \cong \overline{B_1M}$ (a is a bisector).
  iii. $B\hat{M}A \cong B_1\hat{M}A_1$ (a is perpendicular on $\overline{BB_1}$).

  By SAS, $\triangle\, ABM \cong \triangle\, A_1B_1M$, and hence $\overline{AB} \cong \overline{A_1B_1}$.



(b) Clearly $\mathbf{R}_r$ is an isometry and hence it preserves parallelism. Thus, if $a \parallel r$, then $\mathbf{R}_r(a) \parallel \mathbf{R}_r(r)$ then $b \parallel r$ (note that line r is fixed under reflection in r).

Assume now that $b \parallel r$. Since $\mathbf{R}_r(a) = b$, then $\mathbf{R}_r(b) = a$ (this is because $\mathbf{R}_r^{-1} = \mathbf{R}_r$). Thus, if $b \parallel r$, then $\mathbf{R}_r(b) \parallel \mathbf{R}_r(r)$ then $a \parallel r$.

4. (**4 pts. each**)

(a) Let a and b be two lines intersecting in point C with an angle from a to b equals to r. Use the definition of reflection to show that $\mathbf{R}_b \circ \mathbf{R}_a$ is simply $\mathcal{R}_{O,\theta}$, and find the point O and the angle θ.

(b) Let $\triangle\, ABC$ be a triangle with the vertices labelled clockwise such that $\left|\overline{AC}\right| = \left|\overline{BC}\right|$ and $\left|A\hat{C}B\right| = 90°$. Let $\mathbf{R}_{\overleftrightarrow{AB}}$ be the reflection in the line $\overleftrightarrow{AB}$, $\mathbf{R}_{\overleftrightarrow{AC}}$ be the reflection in the line $\overleftrightarrow{AC}$, and $\mathcal{R}_{B,90°}$ be the rotation by 90° counterclockwise around B. Identify the composition $\mathcal{R}_{B,90°} \circ \mathbf{R}_{\overleftrightarrow{AB}} \circ \mathbf{R}_{\overleftrightarrow{AC}}$. Justify your answer.

<u>**Solution**</u>:

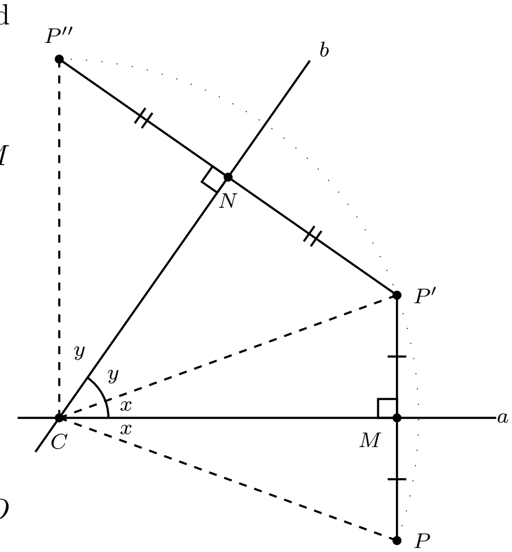
(a) Assume that lines a and line b intersect at point C with a directed angle from a to b equals to $r = (x + y)$.
Let P be a point so that $\mathbf{R}_a(P) = P'$ and $\mathbf{R}_b(P') = P''$.
As it is clear in the diagram, by SAS we have $\triangle\, CPM \cong \triangle\, CP'M$ and $\triangle\, CP'N \cong \triangle\, CP''N$. Hence we have

$$\textcircled{1} \quad \cdots \quad \left|\overline{CP}\right| = \left|\overline{CP'}\right| = \left|\overline{CP''}\right|.$$

Moreover, the (directed) angle from $\overline{CP}$ to $\overline{CP''}$ is

$$\textcircled{2} \quad \cdots \quad 2(x + y) = 2r.$$

Therefore, from ① and ② we have $\mathbf{R}_b \circ \mathbf{R}_a = \mathcal{R}_{C,2r}$. That is O the intersecting points C of lines a and b, and $\theta = 2r$.



(a) Note that $\mathcal{R}_{B,90°}$ is simply $\mathbf{R}_{\overleftrightarrow{BC}} \circ \mathbf{R}_{\overleftrightarrow{AB}}$ with an angle from $\overleftrightarrow{AB}$ to $\overleftrightarrow{BC}$ equals to 45°. That is,

$$\begin{aligned}\mathcal{R}_{B,90°} \circ \left(\mathbf{R}_{\overleftrightarrow{AB}} \circ \mathbf{R}_{\overleftrightarrow{AC}}\right) &= \left(\mathbf{R}_{\overleftrightarrow{BC}} \circ \mathbf{R}_{\overleftrightarrow{AB}}\right) \circ \left(\mathbf{R}_{\overleftrightarrow{AB}} \circ \mathbf{R}_{\overleftrightarrow{AC}}\right) \\ &= \left(\mathbf{R}_{\overleftrightarrow{BC}} \circ \mathbf{R}_{\overleftrightarrow{AC}}\right) = \mathcal{R}_{C,180°}.\end{aligned}$$