Kuwait University
Faculty of Science
Department of Mathematics

Abstract Algebra 0410-262
Final Exam

Thursday, December 20, 2018
Fall 2018/19

Student Name		إسم الطـــالب
Student ID Number		الرقم الجامعي للطـــالب
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Section No. رقم الشعبة	Instructor Name أستاذ المقرر
01 A	**Dr. Abdullah Alazemi**

<u>Instructions to students</u> — تعليمات للطالب

Time allowed: 2 hours. — وقت الإختبار: ساعتين.

This exam contains 5 questions. — يحتوي هذا الإختبار على 5 أسئلة.

ممنوع دخول الآلات الحاسبة أو أي وسيلة للإتصال داخل قاعة الإختبار.
Calculators and communication devices are not allowed in the examination room.

Question 1	/9
Question 2	/9
Question 3	/9
Question 4	/6
Question 5	/10
Total	/43

Good Luck

1. (**3 pts. each**)

(a) Show that if A is an abelian group with addition as the operation, and an operation $*$ is defined on A by $a * b = 0$ for all $a, b \in A$, then $(A, +, *)$ is a commutative ring.

(b) Show that the integral domain $\mathbb{Q}\left(\sqrt{2}\right) = \left\{ a + b\sqrt{2} : a, b \in \mathbb{Q} \right\}$ is a field.

(c) Prove that if R and S are isomorphic rings and R is commutative, then S is commutative as well.

2. (**3+2+2+2 pts.**)

(a) [**The Remainder Theorem**]: Show that if $f(x) \in F[x]$ and $c \in F$ (F is a field), then the remainder in the division of $f(x)$ by $x - c$ is $f(c)$.

(b) Use the remainder theorem to compute the remainder when dividing $f(x) = x^3 - x^2 + x + 3$ by $x - 4$ in $\mathbb{Z}_5[x]$.

(c) Is -1 a root for $f(x) = x^3 - x^2 + x + 3$.

(d) $f(x) = x^3 - x^2 + x + 3$ irreducible over $\mathbb{Z}_5$? Why?

3. (**3 pts. each**)

(a) Express $f(x) = x^3 + x$ as a product of irreducible polynomials in $\mathbb{Z}_5[x]$.

(b) Let $\theta : R \to S$ be a ring homomorphism. Assuming that $\ker\theta$ is a subring of R, show that $\ker\theta$ is an ideal of R.

(c) Let θ be a homomorphism of a ring R with unity e onto a nonzero ring S. Let u be a unit in R. Show that $\theta(u)$ is a unit in S.

4. (**3 pts. each**) Let I and J be two ideals of a ring R, and let $I + J$ be the subring of R defined by $I + J = \{ a + b : a \in I \text{ and } b \in J \}$.

(a) Show that $I + J$ is an ideal of R containing each of I and J.

(b) Show that if $J \subseteq I$, then $I / J = \{ i + J : i \in I \}$ is an ideal of R / J.

5. (**3+1+3+3 pts. each**)

(a) Find all $c \in \mathbb{Z}_3$ such that $\mathbb{Z}_3[x] / (x^2 + c)$ is a field.

(b) Find all $c \in \mathbb{Z}_3$ such that $(x^2 + c)$ is a maximal ideal of $\mathbb{Z}_3[x]$.

(c) Let $\alpha \in \mathbb{Q}[x] / (x^2 - 7)$ be a root of the irreducible polynomial $x^2 - 7 \in \mathbb{Q}[x]$. Express $(1 + \alpha)^2$ in the form $a + b\alpha$ with $a, b \in \mathbb{Q}$.

(d) Let $\mathbb{Z}_2(\alpha) = \{ 0, 1, \alpha, \alpha + 1 \}$ be a field isomorphic to the field $\mathbb{Z}_2[x] / x^2 + x + 1$, where α is a root of $x^2 + x + 1$. Construct the multiplication table of $\mathbb{Z}_2(\alpha)$.