Lecture Notes in Linear Algebra

Dr. Abdullah Al-Azemi

Mathematics Department
Kuwait University

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Chapter

1 Linear Equations and Matrices

1.1 Systems of Linear Equations and Matrices

A general linear system of m equations in the n unknowns $x_1, x_2, \cdots, x_n$:

$$\begin{array}{ccccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & a_{13}x_3 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & a_{23}x_3 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ & & & & & & \vdots & & & & \\ a_{m1}x_1 & + & a_{m2}x_2 & + & a_{m3}x_3 & + & \cdots & + & a_{mn}x_n & = & b_m \end{array} \tag{1.1.1}$$

A system in the Form (1.1.1) is called a **homogeneous** system if $b_1 = b_2 = \cdots = b_m = 0$. Otherwise it is called a **non-homogeneous** system.

A **solution**, if any, of a linear system in n unknowns $x_1, x_2, \cdots, x_n$ is a sequence of n numbers $s_1, s_2, \cdots, s_n$. That is, $x_1 = s_1, x_2 = s_2, \cdots, x_n = s_n$ is a solution satisfying all the above equations.

Any system might have a **unique** solution, **no** solutions, or **infinite** solutions. In general, we say that the system is **consistent** if it has at least one solution, and **inconsistent** if it has no solutions.

Example 1.1.1

Solve the following linear system:

$$\begin{array}{rcrcl} x & + & y & = & 1 \\ 2x & - & y & = & 5 \end{array}$$

Solution:

Clearly adding the two (non-homogeneous) equations, we get $3x = 6$. Thus, $x = 2$ and hence $y = -1$. Therefore, the system has a unique soltion: $x = 2$ and $y = -1$.

Example 1.1.2

Solve the following linear system:

$$\begin{aligned} x + y &= 1 \\ 2x + 2y &= 2 \end{aligned}$$

Solution:

We can eliminate x from the second equation by adding -2 times the first equation to the second. Thus, we get

$$\begin{aligned} x + y &= 1 \\ 0 &= 0 \end{aligned}$$

Thus, we simply can omit the second equation. Therefore, the solutions are of the form $x = 1-y$. Setting a parameter $t \in \mathbb{R}$ for y, we get infinite solutions of the form $x = 1 - t$ and $y = t$. Therefore, the system has infinite solutions.

Example 1.1.3

Solve the following linear system:

$$\begin{aligned} x_1 + x_2 - x_3 &= 1 \\ x_2 - 2x_3 &= 0 \\ 2x_1 + 2x_2 - 2x_3 &= 5 \end{aligned}$$

Solution:

Adding the third equation to -2 times the first equation, we get $0 = 3$ which is impossible. Therefore, this system has no solutions.

Definition 1.1.1

An $m \times n$ **matrix** A is a rectangular array of $m \cdot n$ real numbers arranged in m horizontal rows and n vertical columns. That is,

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1j} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2j} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{i1} & a_{i2} & \dots & a_{ij} & \dots & a_{in} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mj} & \dots & a_{mn} \end{bmatrix}$$

In the matrix A above, we have:

- $a_{11}, a_{12}, \ldots, a_{mn}$ are called **elements** (or **entries**) of the matrix.
- The entry a_{ij} lies in the intersection of row i and column j.
- The **size** (or **order**) of A is m by n, written as $m \times n$.
- The matrix is called a **square** matrix if $m = n$.
- We write $M_{m \times n}$ for the class of all real matrices of size $m \times n$.

If a matrix A is of size $1 \times n$, then we say that A is a **row vector**. In addition, if A is of size $n \times 1$, we say that A is a column vector . In Chapter 3, we speak of n-vectors to be elements of $\mathbb{R}^n$.

Example 1.1.4

Matrix $A = \begin{bmatrix} 2 & 3 & -2 \\ 4 & 1 & -1 \end{bmatrix} \in M_{2\times 3}$ and matrix $B = \begin{bmatrix} 3 & 0 & 1 \\ 4 & 2 & 0 \\ -1 & 4 & 1 \end{bmatrix} \in M_{3\times 3}$.

Matrix Form of Linear System of Equations:

Given a linear system of m equations on n unknowns as in (1.1.1), we transform this system into the matrix form as in (1.1.2)

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix} \tag{1.1.2}$$

We say that the form of (1.1.2) is the **augmented matrix form** . Note that each row in the augmented matrix form correspond to an equation in the associated system.

Example 1.1.5

Here is an example of transforming a system of equations into its augmented matrix form:

$$\begin{array}{ccccccc} x_1 & + & x_2 & - & x_3 = & 1 \\ & & x_2 & - & 2x_3 = & 0 \\ 2x_1 & + & 2x_2 & - & 2x_3 = & 5 \end{array} \quad \to \quad \begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & 1 & -2 & 0 \\ 2 & 2 & -2 & 5 \end{bmatrix}$$

The basic method for solving a linear system is to perform algebraic operations on the system that do not change the solution set so that it produces a simpler version of the same system.

In matrix form, these algebraic operations are called **elementary row operations**:

⋆ Given a matrix $A \in M_{m\times n}$, we define the following **elementary row operations**:

1. interchanging a row by another row,
2. multiplying a row by a non-zero scalar,
3. adding a multiple of a row to another row. That is,

   replace $a_{i1}, a_{i2}, \cdots, a_{in}$ by $a_{i1} + ca_{k1}, a_{i2} + ca_{k2}, \cdots, a_{in} + ca_{kn}$, for $1 \le i, k \le n$.

Example 1.1.6

Use elementary row operations to solve the non-homogeneous system:

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & + & 3x_3 & = & 9 \\ 2x_1 & - & x_2 & + & x_3 & = & 8 \\ 3x_1 & & & - & x_3 & = & 3 \end{array}$$

Solution:

We first transform the system into its augmented matrix form. Then we use the elementary row operations to simplify the form and finally we get the solution in the simpler system.

$$A = \begin{bmatrix} 1 & 2 & 3 & 9 \\ 2 & -1 & 1 & 8 \\ 3 & 0 & -1 & 3 \end{bmatrix} \xrightarrow[r_3-3r_1\to r_3]{r_2-2r_1\to r_2} \begin{bmatrix} 1 & 2 & 3 & 9 \\ 0 & -5 & -5 & -10 \\ 0 & -6 & -10 & -24 \end{bmatrix} \xrightarrow[-\frac{1}{2}r_3\to r_3]{-\frac{1}{5}r_2\to r_2} \begin{bmatrix} 1 & 2 & 3 & 9 \\ 0 & 1 & 1 & 2 \\ 0 & 3 & 5 & 12 \end{bmatrix} \xrightarrow[r_3-3r_2\to r_3]{r_1-2r_2\to r_1}$$

$$\begin{bmatrix} 1 & 0 & 1 & 5 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 2 & 6 \end{bmatrix} \xrightarrow{\frac{1}{2}r_3\to r_3} \begin{bmatrix} 1 & 0 & 1 & 5 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix} \xrightarrow[r_2-r_3\to r_2]{r_1-r_3\to r_1} \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Therefore, the solution is $x_1 = 2$, $x_2 = -1$, and $x_3 = 3$. That is the system is consistent and has a unique solution.

Definition 1.1.2

An $m \times n$ matrix A is called **row equivalent** to an $m \times n$ matrix B if B can be obtained by applying a finite sequence of elementary row operation to A. In this case, we write $A \approx B$.

⋆ **Properties of matrix equivalence**: For any $m \times n$ matrices A, B, and C we have

1. $A \approx A$,
2. $A \approx B \Rightarrow B \approx A$,
3. $A \approx B$ and $B \approx C \quad \Rightarrow \quad A \approx C$.

1.2 Gaussian Elimination

Definition 1.2.1

A matrix $A \in M_{m\times n}$ is said to be in the **reduced row echelon form** (**r.r.e.f.** for short) if it satisfies the following conditions:

1. the row of zeros (if any) should be at the bottom of A,
2. the **leading** (first) entry of a non-zero row should be 1,
3. the leading entry of row $i+1$ should be on the right of the leading entry of row i,
4. the columns that contain a leading entry 1, all its other entries are zeros.

The matrix A is said to be in the **row echelon form** (r.e.f.) if the last condition is not satisfied.

Example 1.2.1

Here are some examples of some matrices to clarify the previous conditions:

(1) $A = \begin{bmatrix} \mathbf{1} & 0 & 0 & 3 & 0 \\ 0 & 0 & \mathbf{1} & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ is r.r.e.f.

(2) $B = \begin{bmatrix} 1 & 3 & 0 & 4 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ 0 & 0 & 1 & -2 \end{bmatrix}$ is not r.r.e.f. since it has a row of zeros in the second row.

(3) $C = \begin{bmatrix} 1 & 0 & 4 & 5 \\ 0 & 2 & -2 & 3 \\ 0 & 0 & 1 & 2 \end{bmatrix}$ is not r.r.e.f. since the leading entry in the second row is not 1.

(4) $D = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 1 & 0 & 0 \end{bmatrix}$ is not r.r.e.f. since the leading entry of the third row is on the left of the leading entry of the second row. Switch 2^{nd} and 3^{rd} rows to get the r.r.e.f. form.

Theorem 1.2.1

Every non-zero $m \times n$ matrix is row equivalent to a unique matrix in the r.r.e.f.

1.2.1 Solving System of Linear Equations

We write O to denote the **zero matrix** whose entries are all 0. We write I_n (or simply) to denote the **identity matrix** with 1's on the main diagonal and zero elsewhere.

In the augmented matrix form, we add a vertical bar | to recognize the scalars in the last column of the matrix. That is a system of linear equations might be transformed into augmented matrix form $\left[\, A \,|\, B \,\right]$. Moreover, the system itself is recognized as $AX = B$, where A is the coefficients, X is the **vector** (one column) of unknowns, and B are the scalars.

Solving $AX = B$

- $AX = B$ where $B \neq O$ (non-homogenous)
  - unique sol
  - no sol
  - infinite sol
- $AX = O$ (homogenous)
  - trivial sol
  - non-trivial sol

Remark 1.2.1

$\star$ **Guass-Jordan method for solving** $AX = B$:

1. Find the r.r.e.f. of augmented matrix $\left[\, A \,|\, B \,\right]$,
2. Solve the reduced system.

Note that, the solution of the reduced system is the solution of the original one.

Theorem 1.2.2

Let $AX = B$ and $CX = D$ be two linear systems of equations. If $[\, A \,|\, B \,] \approx [\, C \,|\, D \,]$, then the two system have the same solutions.

i. Solving Non-Homogenous System $AX = B$ with $B \neq O$

Example 1.2.2

Solve the following system using the Guass-Jordan method.

$$\begin{array}{ccccccc} x & + & 2y & + & 3z & = & 9 \\ 2x & - & y & + & z & = & 8 \\ 3x & & & - & z & = & 3 \end{array}$$

Solution:

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 2 & -1 & 1 & 8 \\ 3 & 0 & -1 & 3 \end{array}\right] \xrightarrow[r_3 - 3r_1 \to r_3]{r_2 - 2r_1 \to r_2} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 0 & -5 & -5 & -10 \\ 0 & -6 & -10 & -24 \end{array}\right] \xrightarrow[-\frac{1}{2}r_3 \to r_3]{-\frac{1}{5}r_2 \to r_2} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 0 & 1 & 1 & 2 \\ 0 & 3 & 5 & 12 \end{array}\right] \xrightarrow[r_3 - 3r_2 \to r_3]{r_1 - 2r_2 \to r_1}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 5 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 2 & 6 \end{array}\right] \xrightarrow{\frac{1}{2}r_3 \to r_3} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 5 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right] \xrightarrow[r_2 - r_3 \to r_2]{r_1 - r_3 \to r_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Therefore, the reduced system is: $x = 2$, $y = -1$, and $z = 3$. Thus, $X = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$ is a **unique solution** to the system.

Remark 1.2.2

The system $AX = B$ has a unique solution if and only if $A \approx I$.

Example 1.2.3

Solve the following system using the Guass-Jordan method.

$$\begin{array}{ccccccccc} x_1 & + & x_2 & - & x_3 & + & 4x_4 & = & 1 \\ & + & x_2 & - & 3x_3 & + & 4x_4 & = & 0 \\ 2x_1 & + & 2x_2 & - & 2x_3 & + & 8x_4 & = & 2 \end{array}$$

Solution:

$$\left[\begin{array}{cccc|c} 1 & 1 & -1 & 4 & 1 \\ 0 & 1 & -3 & 4 & 0 \\ 2 & 2 & -2 & 8 & 2 \end{array}\right] \xrightarrow{r_3-2r_1\to r_3} \left[\begin{array}{cccc|c} 1 & 1 & -1 & 4 & 1 \\ 0 & 1 & -3 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{r_1-r_2\to r_1} \left[\begin{array}{cccc|c} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & -3 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Therefor, the reduced system is:

$$\begin{aligned} x_1 + 2x_3 &= 1 \\ x_2 - 3x_3 + 4x_4 &= 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} x_1 &= 1 - 2x_3 \\ x_2 &= 3x_3 - 4x_4 \end{aligned}$$

We now fix $x_3 = r$ and $x_4 = t$ for $r, t \in \mathbb{R}$ to get the following **infinite many solutions**:

$$\begin{aligned} x_1 &= 1 - 2r \\ x_2 &= 3r - 4t \end{aligned} \quad \Rightarrow \quad X = \begin{bmatrix} 1-2r \\ 3r-4t \\ r \\ t \end{bmatrix} \text{ for all } r, t \in \mathbb{R}.$$

Remark 1.2.3

The system $AX = B$ has infinity many solutions if the number of unknowns (columns of matrix A) is more than the number of equations (rows of matrix A).

Example 1.2.4

Solve the following system using the Guass-Jordan method.

$$\begin{array}{ccccccccc} x_1 & + & x_2 & - & x_3 & + & 4x_4 & = & 1 \\ & + & x_2 & - & 3x_3 & + & 4x_4 & = & 0 \\ 2x_1 & + & 2x_2 & - & 2x_3 & + & 8x_4 & = & 3 \end{array}$$

Solution:

$$\left[\begin{array}{cccc|c} 1 & 1 & -1 & 4 & 1 \\ 0 & 1 & -3 & 4 & 0 \\ 2 & 2 & -2 & 8 & 2 \end{array}\right] \xrightarrow{r_3-2r_1\to r_3} \left[\begin{array}{cccc|c} 1 & 1 & -1 & 4 & 1 \\ 0 & 1 & -3 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

At this point, it can be seen that the third row suggests that $0 = 1$ which is not possible. Therefore, this system has **no solution**.

Remark 1.2.4

The system $AX = B$ has no solution whenever $A \approx$ a matrix with a row of zeros while $[\,A \,|\, B\,] \approx$ a matrix with no rows of zeros.

ii. **Solving Homogenous System** $AX = O$

Example 1.2.5

Solve the following system using the Guass-Jordan method.

$$\begin{array}{rcrcrcr} x & + & 2y & + & 3z & = & 0 \\ x & + & 3y & + & 2z & = & 0 \\ 2x & + & y & - & 2z & = & 0 \end{array}$$

Solution:

$$\left[\begin{array}{rrr|r} 1 & 2 & 3 & 0 \\ 1 & 3 & 2 & 0 \\ 2 & 1 & -2 & 0 \end{array}\right] \xrightarrow[r_3-2r_1\to r_3]{r_2-r_1\to r_2} \left[\begin{array}{rrr|r} 1 & 2 & 3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -3 & -8 & 0 \end{array}\right] \xrightarrow[r_3+3r_2\to r_3]{r_1-2r_2\to r_1} \left[\begin{array}{rrr|r} 1 & 0 & 5 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -11 & 0 \end{array}\right] \xrightarrow{\frac{-1}{11}r_3\to r_3}$$

$$\left[\begin{array}{rrr|r} 1 & 0 & 5 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow[r_2+r_3\to r_2]{r_1-5r_3\to r_1} \left[\begin{array}{rrr|r} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Therefore, the reduced system is: $x = 0$, $y = 0$, and $z = 0$. Thus, $X = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is a **trivial solution**.

Remark 1.2.5

The system $AX = O$ has a trivial solution whenever $A \approx I$.

Example 1.2.6

Solve the following system using the Guass-Jordan method.

$$\begin{array}{ccccccccc} x_1 & + & x_2 & - & x_3 & + & 4x_4 & = & 0 \\ & + & x_2 & - & 3x_3 & + & 4x_4 & = & 0 \\ 2x_1 & + & 2x_2 & - & 2x_3 & + & 8x_4 & = & 0 \end{array}$$

Solution:

$$\left[\begin{array}{cccc|c} 1 & 1 & -1 & 4 & 0 \\ 0 & 1 & -3 & 4 & 0 \\ 2 & 2 & -2 & 8 & 0 \end{array}\right] \xrightarrow{r_3-2r_1\to r_3} \left[\begin{array}{cccc|c} 1 & 1 & -1 & 4 & 0 \\ 0 & 1 & -3 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{r_1-r_2\to r_1} \left[\begin{array}{cccc|c} 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & -3 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Therefor, the reduced system is:

$$\begin{array}{rcl} x_1 + 2x_3 & = & 0 \\ x_2 - 3x_3 + 4x_4 & = & 0 \end{array} \Rightarrow \begin{array}{rcr} x_1 & = & -2x_3 \\ x_2 & = & 3x_3 - 4x_4 \end{array}$$

We now fix $x_3 = r$ and $x_4 = t$ for $r, t \in \mathbb{R}$ to get the following **non-trivial solution**:

$$\begin{array}{rcr} x_1 & = & -2r \\ x_2 & = & 3r - 4t \end{array} \Rightarrow X = \begin{bmatrix} -2r \\ 3r - 4t \\ r \\ t \end{bmatrix} \text{ for all } r, t \in \mathbb{R}.$$

Remark 1.2.6

The system $AX = O$ has a non-trivial solution if the number of unknowns (columns of matrix A) is greater than the number of equations (rows of matrix A) in the reduced system.

Exercise 1.2.1

1. Find the reduced row echelon form (r.r.e.f.) of the following matrix:

$$\begin{bmatrix} 2 & 4 & 6 & 0 \\ 1 & 2 & 4 & 0 \\ 1 & 3 & 3 & 1 \end{bmatrix}$$

2. Solve the systems:
$$\begin{array}{rcrcrcr} x & + & y & + & z & = & 0 \\ x & + & 2y & + & 3z & = & 0 \\ x & + & 3y & + & 4z & = & 0 \\ x & + & 4y & + & 5z & = & 0. \end{array}$$

3. Solve the systems
$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & - & 3x_3 & = & 6 \\ 2x_1 & - & x_2 & + & 4x_3 & = & 1 \\ x_1 & - & x_2 & + & x_3 & = & 3. \end{array}$$

1.3 Matrix Operations

Let A be a given $m \times n$ matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}.$$

Then we might write $A = [a_{ij}]$ where a_{ij} correspond to the entry at row i and column j. We also might write $(A)_{ij}$ to denote the same entry a_{ij}.

We also use the terms: **row vector**, $\text{row}_i(A)$, and **column vector**, col_j, to denote the i^{th} row and j^{th} column of the matrix A. That is

$$\text{row}_i(A) = \begin{bmatrix} a_{i1} & a_{i2} & \cdots & a_{in} \end{bmatrix}_{1\times n} \qquad \text{and} \qquad \text{col}_j(A) = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix}_{m\times 1}.$$

An $n \times n$ matrix is called a **square matrix**. Moreover, the entries $a_{11}, a_{22}, \cdots, a_{nn}$ are said to be on the **main diagonal** of A.

The **trace of** A, denoted by $tr(A)$, is the sum of the entries on the main diagonal. If A is not a square matrix, then the trace of A is undefined.

Remark 1.3.1

Two $m \times n$ matrices A and B are said to be **equal** if $(A)_{ij} = (B)_{ij}$ for all $1 \leq i \leq m$ and $1 \leq j \leq n$.

Example 1.3.1

Find the values of $a, b, c,$ and d if

$$\begin{bmatrix} a+b & c+d \\ c-d & a-b \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}.$$

Solution:

Since the two matrices are equal. Then

$$\left.\begin{aligned} a+b&=1 \\ a-b&=1 \end{aligned}\right\}+ \implies 2a=2 \implies a=1 \text{ and hence } b=0.$$

$$\left.\begin{aligned} c+d&=2 \\ c-d&=-2 \end{aligned}\right\}+ \implies 2c=0 \implies c=0 \text{ and hence } d=2.$$

- **Matrix transpose**

Definition 1.3.1

If $A=[a_{ij}]$ is an $m \times n$ matrix, then the $n \times m$ matrix $A^T=[a_{ji}]$, where $1 \leq j \leq n$ and $1 \leq i \leq m$, is called the **transpose of** A. Observe that A^T is resulted by interchanging rows and columns of A.

Example 1.3.2

Here are some examples of matrices and their transpose:

$$A=\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}_{3\times 3}, \quad B=\begin{bmatrix} 1 & 4 & 0 \\ -1 & 2 & 3 \end{bmatrix}_{2\times 3}, \text{ and } \quad C=\begin{bmatrix} 2 & 5 & 7 \end{bmatrix}_{1\times 3}$$

$$A^T=\begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix}_{3\times 3}, \quad B^T=\begin{bmatrix} 1 & -1 \\ 4 & 2 \\ 0 & 3 \end{bmatrix}_{3\times 2}, \text{ and } \quad C^T=\begin{bmatrix} 2 \\ 5 \\ 7 \end{bmatrix}_{3\times 1}$$

- **Matrix addition and substraction**

To add two $m \times n$ matrices $A = [a_{ij}]$ and $B = [b_{ij}]$, we must have the size(A) = size(B). Then, $A \pm B = C$, where $C = [c_{ij}]$ with $c_{ij} = a_{ij} \pm b_{ij}$, for all $1 \leq i \leq m$ and $1 \leq j \leq n$. That is,

$$(A \pm B)_{ij} = (A)_{ij} \pm (B)_{ij} = a_{ij} \pm b_{ij}.$$

Example 1.3.3

If possible, find $A + B$ and $A^T - B$, where

$$A = \begin{bmatrix} 2 & 1 \\ 3 & 0 \\ -1 & 1 \end{bmatrix}_{3\times 2}, \text{ and } B = \begin{bmatrix} 1 & 2 & 4 \\ 5 & 1 & 1 \end{bmatrix}_{2\times 3}.$$

Solution:

Clearly, $A+B$ is not possible as they have different sizes. On the other hand, size(A^T) = size(B), and

$$A^T - B = \begin{bmatrix} 2 & 3 & -1 \\ 1 & 0 & 1 \end{bmatrix}_{2\times 3} - \begin{bmatrix} 1 & 2 & 4 \\ 5 & 1 & 1 \end{bmatrix}_{2\times 3} = \begin{bmatrix} 1 & 1 & -5 \\ -4 & -1 & 0 \end{bmatrix}_{2\times 3}.$$

- **Matrix multiplication**

If $A = [a_{ij}]$ is any $m \times n$ matrix and c is any scalar, then $c\,A = [c\,a_{ij}]$ for all $1 \leq i \leq m$ and $1 \leq j \leq n$. That is, $(cA)_{ij} = c(A)_{ij} = c\,a_{ij}$. The matrix $c\,A$ is called a **scalar multiple** of A.

Definition 1.3.2

An n-vector $X = (x_1 \;\; x_2 \;\; \cdots \;\; x_n)$ can be written as $X = (x_1, x_2, \cdots, x_n)$. The **dot (inner) product** of the n-vectors $X = (x_1, x_2, \cdots, x_n)$ and $Y = (y_1, y_2, \cdots, y_n)$ is defined by

$$X \cdot Y = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n = \sum_{i=1}^{n} x_i y_i. \tag{1.3.1}$$

Example 1.3.4

If $X = (1, 0, 2, -1)$ and $Y = (3, 5, -1, 4)$, then $X \cdot Y = 3 + 0 + (-2) + (-4) = -3$.

Definition 1.3.3

Let $A = [a_{ij}]$ be an $m \times p$ matrix and $B = [b_{ij}]$ be a $p \times n$ matrix. Then, the product of A and B is the $m \times n$ matrix $AB = [(AB)_{ij}]$, where $(AB)_{ij} = \text{row}_i(A) \cdot \text{col}_j(B)$. That is,

$$(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ip}b_{pj} = \sum_{k=1}^{p} a_{ik}b_{kj}, \text{ for } 1 \leq i \leq m \text{ and } 1 \leq j \leq n.$$

The product is undefined if the number of columns of A not equals the number of rows of B.

Example 1.3.5

If $A = \begin{bmatrix} 1 & 3 & -1 \\ 2 & 0 & 1 \end{bmatrix}_{2\times 3}$, and $B = \begin{bmatrix} 2 & 1 \\ 0 & 2 \\ -1 & 2 \end{bmatrix}_{3\times 2}$, then $AB = \begin{bmatrix} 2+0+1 & 1+6-2 \\ 4+0-1 & 2+0+2 \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ 3 & 4 \end{bmatrix}_{2\times 2}$.

Example 1.3.6

Find AB and A^TB if possible, where

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 1 & -2 & 0 \end{bmatrix}, \text{ and } B = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 5 & 1 \end{bmatrix}.$$

Solution:

- AB is not defined # columns in A is not the same as # rows of B.
- $A^TB = \begin{bmatrix} 1 & 1 \\ 0 & -2 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 4 & 1 & -1 \\ 2 & 5 & 1 \end{bmatrix} = \begin{bmatrix} 4+2 & 1+5 & -1+1 \\ -4 & -10 & -2 \\ 8 & 2 & -2 \end{bmatrix} = \begin{bmatrix} 6 & 6 & 0 \\ -4 & -10 & -2 \\ 8 & 2 & -2 \end{bmatrix}$

Remark 1.3.2

In general, $AB \neq BA$. For instance, consider $A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$. Check it yourself!!

It is always true that $AB = BA$ if $A = I_n$ or that $A = B$.

Example 1.3.7

Let $A = \begin{bmatrix} 1 & -2 & 0 \end{bmatrix}$. Find all values of c so that $c\,A \cdot A^T = 15$.

Solution:

Note that $cA \cdot A^T = c \begin{bmatrix} 1 & -2 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} = 5c$. Therefore, $5c = 15$ and hence $c = 3$.

Example 1.3.8

Let $A = \begin{bmatrix} 1 & -2 & 3 \\ 4 & 2 & 1 \\ 0 & 1 & -2 \end{bmatrix}$, and $B = \begin{bmatrix} 1 & 4 \\ 3 & -1 \\ -2 & 2 \end{bmatrix}$. Compute the $(3, 2)$-entry, $(AB)_{32}$, of AB.

Solution:

$(AB)_{32} = \text{row}_3(A) \cdot \text{col}_2(B) = [0 \;\; 1 \;\; -2] \cdot [4 \;\; -1 \;\; 2]^T = -1 - 4 = -5.$

Theorem 1.3.1

If A is an $m \times p$ matrix and B is a $p \times n$ matrix, then

(a) for each $1 \leq j \leq n$, $\text{col}_j(AB) = A\ \text{col}_j(B)$.

(b) for each $1 \leq i \leq m$, $\text{row}_i(AB) = \big(\text{row}_i(A)\big)\ B$.

<u>TRUE or FALSE</u>:

$\star$ If a matrix B has a column of zeros, then the product AB has a column of zeros as well. (TRUE).

reason: Assume that column j of B is a column of zero. Then, $\text{col}_j(AB) = A\,\text{col}_j(B) = \mathbf{0}$.

Example 1.3.9

Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ -1 & 5 \end{bmatrix}_{3\times 2}$ and $B = \begin{bmatrix} -2 & 3 & 4 \\ 3 & 2 & 1 \end{bmatrix}_{2\times 3}$.

1. Find the first column of AB.
2. Find the second row of AB.

Solution:

We simply use Theorem 1.3.1 as follows:

1. $col_1(AB) = A\, col_1(B) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ -1 & 5 \end{bmatrix} \begin{bmatrix} -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ 17 \end{bmatrix}$
2. $row_2(AB) = row_2(A)\, B = \begin{bmatrix} 3 & 4 \end{bmatrix} \begin{bmatrix} -2 & 3 & 4 \\ 3 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 17 & 16 \end{bmatrix}$

Definition 1.3.4

Let A be an $m \times n$ matrix and $X \in \mathbb{R}^n$ (be an n-vector), then

$$AX = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix}$$

$$= x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} = x_1 \mathrm{col}_1(A) + x_2 \mathrm{col}_2(A) + \cdots + x_n \mathrm{col}_n(A).$$

That is, AX is a **linear combination** of columns of A and the entries of X are the coefficients.

Example 1.3.10

Write the following product as a linear combination of the columns of the first matrix.

$$\begin{bmatrix} 1 & 3 \\ 2 & 1 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} + (-2) \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}.$$

Example 1.3.11

Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ -1 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 3 & 4 \\ 3 & 2 & 1 \end{bmatrix}$. Find the second column of AB as a linear combination of columns of A.

Solution:

$$\mathrm{col}_2(AB) = A\,\mathrm{col}_2(B) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ -1 & 5 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} + 2 \begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix}$$

Example 1.3.12

If A and B are $n \times n$ matrices, then

1. $tr(cA) = c\,tr(A)$, where c is any real number.
2. $tr(AB) = tr(BA)$.

Solution:

1. $tr(cA) = c\,(A)_{11} + c\,(A)_{22} + \cdots + c\,(A)_{nn} = c[(A)_{11} + (A)_{22} + \cdots + (A)_{nn}] = c\,tr(A)$.
2.

$$tr(AB) = \sum_{i=1}^{n} (AB)_{ii} = \sum_{i=1}^{n} \sum_{j=1}^{n} (A)_{ij}(B)_{ji} = \sum_{j=1}^{n} \sum_{i=1}^{n} (B)_{ji}(A)_{ij} = \sum_{j=1}^{n} (BA)_{jj} = tr(BA).$$

Exercise 1.3.1

1. Let $A = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$. Compute $A^2 + I_2$.
2. Let $A = \begin{bmatrix} 1 & 2 \\ 4 & 5 \\ 3 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 & -3 \\ -3 & 1 & 4 \end{bmatrix}$. Find the third row of AB.
3. Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \\ 2 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 5 & 1 \end{bmatrix}$. Find AB and express the second column of AB as a linear combination of the columns of A.
4. Let $A = \begin{bmatrix} 1 & 2 & 7 \\ 1 & -5 & 7 \\ 0 & -1 & 10 \end{bmatrix}$. Find $tr(A)$.
5. Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Find A^{1977}, and find all matrices B such that $AB = BA$.
6. Let $A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$. Find A^{100}.
7. Find a 2×2 matrix $B \neq O$ and $B \neq I_2$ so that $AB = BA$ if $A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$
8. Show that if A and B are $n \times n$ matrices, then $tr(A + B) = tr(A) + tr(B)$.
9. Show that there are no 2×2 matrices A and B so that $AB - BA = I_2$.

1.4 Inverses; Algebraic Properties of Matrices

Theorem 1.4.1: Properties of Matrix Arithmetic

Let A, B, and C be matrices of appropriate sizes, and let $r, s \in \mathbb{R}$. Then:

1. $A + B = B + A$ **(Commutative law for matrix addition)**
2. $A + (B + C) = (A + B) + C$ **(Associative law for matrix addition)**
3. $A(BC) = (AB)C$ **(Associative law for matrix multiplication)**
4. $A(B + C) = AB + AC$ **(Left distributive law)**
5. $(A + B)C = AC + BC$ **(right distributive law)**
6. $A(B - C) = AB - AC$
7. $(A - B)C = AC - BC$
8. $r(A + B) = rA + rB$
9. $r(A - B) = rA - rB$
10. $(r + s)A = rA + sA$
11. $(r - s)A = rA - sA$
12. $r(sA) = (rs)A$
13. $r(AB) = (rA)B = A(rB)$

Proof:

We only proof (1) and (10). Let $A = [a_{ij}]$ and $B = [b_{ij}]$ for $1 \leq i \leq m$ and $1 \leq j \leq n$. Then

1. $A + B = [a_{ij} + b_{ij}] = [b_{ij} + a_{ij}] = B + A$.
10. $(r + s)A = (r + s)[a_{ij}] = [(r + s)\, a_{ij}] = [r\, a_{ij} + s\, a_{ij}] = [r\, a_{ij}] + [s\, a_{ij}] = rA + sA$.

Theorem 1.4.2: Properties of Zero Matrices

Let c be a scalar, and let A be a matrix of an appropriate size. Then:

1. $A + O = O + A = A$
2. $A - O = A$
3. $A - A = A + (-A) = O$

4. $OA = O$
5. If $cA = O$, then $c = 0$ or $A = O$.

Proof:

We only proof (5). Let $A = [a_{ij}]$ for $1 \le i \le m$ and $1 \le j \le n$. If $cA = O$, then for each i and j, we have $(cA)_{ij} = c(A)_{ij} = c\,a_{ij} = 0$. Therefore, either $c = 0$ or we have $a_{ij} = 0$ for all i and j. Therefore, either $c = 0$ or $A = O$.

Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix}$, $C = \begin{bmatrix} 2 & 5 \\ 1 & 2 \end{bmatrix}$, and $D = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$. Then, $AB = AC = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$, but $B \ne C$. That is, the cancellation law does not hold here.

For any $a, b \in \mathbb{R}$, we have $ab = 0$ implies that $a = 0$ or $b = 0$. However, in matrices we have $AD = O$ but $A \ne O$ and $D \ne O$.

Moreover, the $n \times n$ identity matrix I commutes with any other matrix. That is, $AI = IA = A$ for any $n \times n$ matrix A.

Definition 1.4.1

A matrix $A \in M_{n\times n}$ is called **nonsingular** or **invertable** if there exists a matrix $B \in M_{n\times n}$ such that

$$A\,B = B\,A = I_n.$$

In particular, B is called **the inverse of** A and is denoted by A^{-1}. If there is no such B, we say that A is **singular** which means that A has no inverse.

Example 1.4.1

Here is an example of a 2×2 matrix along with its inverse.

$$\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2.$$

Theorem 1.4.3

If a matrix has an inverse, then its inverse is unique.

Proof:

Assume that $A \in M_{n\times n}$ is a matrix with two inverses B and C, then

$$B = B\,I_n = B\,(A\,C) = (B\,A)\,C = I_n\,C = C.$$

Therefore, A has a unique inverse.

Theorem 1.4.4

The matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is nonsingular iff $ad - bc \neq 0$, in which case the inverse of A is

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Proof:

We simply show that $AA^{-1} = A^{-1}A = I$. Here, we only do the following:

$$\begin{aligned} A\,A^{-1} &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{ad-bc}\begin{bmatrix} ad-bc & -ab+ab \\ cd-cd & -bc+ad \end{bmatrix} \\ &= \frac{1}{ad-bc}\begin{bmatrix} ad-bc & 0 \\ 0 & -bc+ad \end{bmatrix} = I. \end{aligned}$$

We note that the quantity $ad-bc$ above is called the **determinant** of the 2×2 matrix A and is denoted by $\det(A) = ad - bc$. The determinant will be discussed in general sizes in Chapter 2. As an example, if $A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$, then $\det(A) = 3 - 2 = 1$ and hence the inverse of A is $A^{-1} = \frac{1}{3-2}\begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$.

Theorem 1.4.5

If A and B are nonsingular $n \times n$ matrices, then $A\,B$ is nonsingular and $(A\,B)^{-1} = B^{-1}\,A^{-1}$.

Proof:

Since both A and B are nonsingular, then both A^{-1} and B^{-1} exist. Thus

$$A\,B\,(B^{-1}\,A^{-1}) = A\,(B\,B^{-1})\,A^{-1} = A\,I\,A^{-1} = A\,A^{-1} = I.$$

$$A\,B\,(B^{-1}\,A^{-1}) = A\,(B\,B^{-1})\,A^{-1} = A\,I\,A^{-1} = A\,A^{-1} = I.$$

Thus, $A\,B$ is nonsingular and $(A\,B)^{-1} = B^{-1}\,A^{-1}$.

Remark 1.4.1

Let m and n be nonnegative integers and let A and B be two matrices of appropriate sizes, then

1. $A^0 = I$ and $A^n = A\,A\,\cdots\,A$ (n-times),
2. $A^m A^n = A^{m+n}$,
3. $(A^m)^n = A^{mn}$,
4. $(AB)^n = (AB)(AB)\cdots(AB)$, ($n$-times), and in general $(AB)^n \neq A^n B^n$.
5. In general, $(A+B)^2 = (A+B)(A+B) = A^2 + AB + BA + B^2 \neq A^2 + 2AB + B^2$.

Theorem 1.4.6

If A is a nonsingular matrix and n is a nonnegative integer, then:

1. A^{-1} is nonsingular and $\left(A^{-1}\right)^{-1} = A$.
2. A^n is nonsingular and $\left(A^n\right)^{-1} = A^{-n} = (A^{-1})^n$.
3. $k\,A$ is nonsingular for any nonzero scalar k, and $\left(k\,A\right)^{-1} = k^{-1}A^{-1}$.

Proof:

We use the idea of the definition. A matrix B is the inverse of matrix C if $BC = CB = I$.

1. Clearly, $A^{-1}\,A = A\,A^{-1} = I$ and hence A is the inverse of A^{-1}.
2. $A^n\,A^{-n} = A^{n-n} = A^0 = I$. That is A^{-n} is the inverse of A^n.
3. $(kA)(k^{-1}A^{-1}) = (kk^{-1})(AA^{-1}) = I$. Also, $(k^{-1}A^{-1})(kA) = I$. Thus, $\left(kA\right)^{-1} = k^{-1}A^{-1}$.

Theorem 1.4.7

Let A and B be two matrices of appropriate size and let c be a scalar. Then:

1. $(A^T)^T = A$,
2. $(A \pm B)^T = A^T \pm B^T$,
3. $(c\,A)^T = c\,A^T$, and
4. $(AB)^T = B^T A^T$. This result can be extended to three or more factors.

Exercise 1.4.1

If A is a square matrix and n is a positive integer, is it true that $(A^n)^T = (A^T)^n$? Justify your answer.

Theorem 1.4.8

If A is nonsingular matrix, then A^T is also nonsingular and $(A^T)^{-1} = (A^{-1})^T$.

Proof:

We simply show that the product $A^T\,(A^{-1})^T = (A^{-1})^T A^T = I$. Clearly, $A^T(A^{-1})^T = (A^{-1}\,A)^T = I^T = I$. Similarly, $(A^{-1})^T A^T = I$.

Example 1.4.2

Let $A = \begin{pmatrix} 2 & 0 \\ 1 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 4 \\ 3 & 5 \end{pmatrix}$. Find AB, and $B^T A^T$.

Solution:

Clearly, $AB = \begin{pmatrix} 2+0 & 8+0 \\ 1-3 & 4-5 \end{pmatrix} = \begin{pmatrix} 2 & 8 \\ -2 & -1 \end{pmatrix}$. Moreover, $B^T A^T = (AB)^T = \begin{pmatrix} 2 & -2 \\ 8 & -1 \end{pmatrix}$

Definition 1.4.2

Let A be an $n \times n$ (square) matrix and $p(x) = a_0 + a_1x + \cdots + a_mx^m$ be any polynomial. Then we define the **matrix polynomial in** A as the $n \times n$ matrix $p(A)$ where

$$p(A) = a_0I_n + a_1A + \cdots + a_mA^m.$$

Example 1.4.3

Compute $p(A)$ where $p(x) = x^2 - 2x - 3$ and $A = \begin{bmatrix} -1 & 2 \\ 0 & 3 \end{bmatrix}$.

Solution:

$$\begin{aligned} p(A) = A^2 - 2A - 3I_2 &= \begin{bmatrix} -1 & 2 \\ 0 & 3 \end{bmatrix}^2 - 2\begin{bmatrix} -1 & 2 \\ 0 & 3 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 4 \\ 0 & 9 \end{bmatrix} - \begin{bmatrix} -2 & 4 \\ 0 & 6 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O \end{aligned}$$

Example 1.4.4

Show that if $A^2 + 5A - 2I = O$, then $A^{-1} = \frac{1}{2}(A + 5I)$.

Solution:

$A^2 + 5A - 2I = O$ implies that $2I = A^2 + 5A = A(A + 5I)$. Then $I = \frac{1}{2}A(A + 5I)$. Therefore, $I = A\left(\frac{1}{2}(A + 5I)\right)$, which shows that $A^{-1} = \frac{1}{2}(A + 5I)$.

Example 1.4.5

Show that if A is a square matrix such that $A^k = O$ for some positive integer k, then the matrix A is nonsingular and $(I - A)^{-1} = I + A + A^2 + \cdots + A^{k-1}$.

Solution:

$$
\begin{aligned}
(I - A)(I + A + \cdots + A^{k-1}) &= \left[I + \cancel{A} + \cancel{A^2} + \cdots + \cancel{A^{k-1}}\right] - \left[\cancel{A} + \cancel{A^2} + \cancel{A^3} + \cdots + \cancel{A^{k-1}} + A^k\right] \\
&= I - A^k = I - O = I.
\end{aligned}
$$

Exercise 1.4.2

1. Let $A = \begin{bmatrix} 1 \\ 3 \\ -1 \\ 3 \end{bmatrix}$. Find all constants $c \in \mathbb{R}$ such that $(cA)^T \cdot (cA) = 5$.

2. Let A be an $n \times n$ matrix such that $A^3 = O$. With justification, prove or disprove that
$$(I_n - A)^{-1} = A^2 + A + I_n.$$

3. A square matrix A is said to be **idempotent** if $A^2 = A$.
   (a) Show that if A is idempotent, then so is $I - A$.
   (b) Show that if A is idempotent, then $2A - I$ is nonsingular and is its own inverse.

4. Let $p_1(x) = x^2 - 9$, $p_2(x) = x + 3$, and $p_3(x) = x - 3$. Show that $p_1(A) = p_2(A)p_3(A)$ for any square matrix A.

5. Let A be a square matrix. Then
   (a) Show that $(I - A)^{-1} = I + A + A^2 + A^3$ if $A^4 = O$.
   (b) Show that $(I - A)^{-1} = I + A + A^2 + \cdots + A^n$ if $A^{n+1} = O$.

6. Let J_n be the $n \times n$ matrix each of whose entries is 1. Show that if $n > 1$, then
$$(I - J_n)^{-1} = I - \frac{1}{n-1} J_n.$$

7. Let A, B, and C be $n \times n$ matrices such that $D = AB + AC$ is non-singular.
   (a) Find A^{-1} if possible.
   (b) Find $(B + C)^{-1}$ if possible.

8. Let $A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$, and $B = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 2 & 0 \end{bmatrix}$. Find C if $(B^T + C)A^{-1} = B^T$.

9. Let $A^{-1} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \\ 2 & 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$. Find C if $AC = B^T$.
   [**Hint**: Consider multiplying both sides from the left with A^{-1}.]

1.5 A Method for Finding A^{-1}

Unless otherwise specified, all matrices in this section are considered to be square matrices.

Theorem 1.5.1

If A is an $n \times n$ matrix, then the following statements are equivalent:

1. A is nonsingular.
2. $AX = O$ has only the trivial solution.
3. A is row equivalent to I_n.

Remark 1.5.1

⋆ **How to find A^{-1} for a given $A \in M_{n\times n}$:**

1. form the augmented matrix $[\,A\,|\,I_n\,]$,
2. find the r.r.e.f. of $[\,A\,|\,I_n\,]$, say $[\,C\,|\,B\,]$:

   (a) if $C = I_n$, then A is non-singular and $A^{-1} = B$,
   (b) if $C \neq I_n$, then C has a row of zeros and A is singular matrix with no inverse.

Example 1.5.1

Find, if possible, A^{-1} for $A = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$

Solution:

$$\left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 2 & 3 & 0 & 1 \end{array}\right] \xrightarrow{r_2 - 2r_1 \to r_2} \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{array}\right] \xrightarrow{r_1 - r_2 \to r_1} \left[\begin{array}{cc|cc} 1 & 0 & 3 & -1 \\ 0 & 1 & -2 & 1 \end{array}\right].$$

Therefore, A is non-singular and $A^{-1} = \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix}$.

Example 1.5.2

Find, if possible, A^{-1} for $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 2 & 0 & 1 \end{bmatrix}$

Solution:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 2 & 0 & 1 & 0 \\ 2 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[r_3 - 2r_1 \to r_3]{r_2 - r_1 \to r_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & -1 & -2 & 0 & 1 \end{array}\right] \xrightarrow{-r_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 & -1 \end{array}\right]$$

$$\xrightarrow[r_2 - r_3 \to r_2]{r_1 - r_3 \to r_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & -3 & 1 & 1 \\ 0 & 0 & 1 & 2 & 0 & -1 \end{array}\right].$$ Therefore, A is non-singular and $A^{-1} = \begin{bmatrix} -1 & 0 & 1 \\ -3 & 1 & 1 \\ 2 & 0 & -1 \end{bmatrix}$.

Example 1.5.3

Find, if possible, A^{-1} for $A = \begin{bmatrix} 1 & 2 & -3 \\ 1 & -2 & 1 \\ 5 & -2 & -3 \end{bmatrix}$

Solution:

$$\left[\begin{array}{ccc|ccc} 1 & 2 & -3 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 & 1 & 0 \\ 5 & -2 & -3 & 0 & 0 & 1 \end{array}\right] \xrightarrow[r_3 - 5r_1 \to r_3]{r_2 - r_1 \to r_2} \left[\begin{array}{ccc|ccc} 1 & 2 & -3 & 1 & 0 & 0 \\ 0 & -4 & 4 & -1 & 1 & 0 \\ 0 & -12 & 12 & -5 & 0 & 1 \end{array}\right] \xrightarrow{r_3 - 3r_2 \to r_3}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & -3 & 1 & 0 & 0 \\ 0 & -4 & 4 & -1 & 1 & 0 \\ 0 & 0 & 0 & -2 & -3 & 1 \end{array}\right].$$

At this point, we can conclude that this matrix is singular with no inverse because of the fact that the third row in the first part is a zero row. In particular, A^{-1} does not exist.

Remark 1.5.2

Let A be an $n \times n$ matrix:

1. if A is row equivalent to I_n, then it is non-singular,

Example 1.5.4

$A = \begin{bmatrix} -3 & 0 \\ 0 & 5 \end{bmatrix}$ is row equivalent to I_2.

2. if A is row equivalent to a matrix with a row of zeros, then it is singular.

Example 1.5.5

$A = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}$ is row equivalent to a matrix with a row of zeros.

TRUE or FALSE:

⋆ If A and B are singular matrices, then so is $A + B$. (FALSE).

reason: $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ are two singular matrices, while $A + B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is a non-singular.

⋆ If A and B are non-singular matrices, then so is $A + B$. (FALSE).

reason: $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ are two non-singular matrices, while $A + B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is a singular.

Exercise 1.5.1

1. Let $A^T = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$. Find $(2A)^{-1}$.

   [**Hint**: If c is a nonzero constant, then what is $(cA)^{-1}$?]

2. Let $A^{-1} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$. Find all $x, y, z \in \mathbb{R}$ such that $\begin{bmatrix} x & y & z \end{bmatrix} A = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$.

3. Let $A = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 3 & 0 \\ 1 & 1 & -1 \end{bmatrix}$.

   (a) Find B^{-1}.

   (b) Find C if $A = BC$.

1.6 More on Linear Systems and Invertible Matrices

Theorem 1.6.1

A (nonhomogeneous) system of linear equations has zero, one, or infinitely many solutions.

Proof:

If $AX = B$, $B \neq O$ has no solutions or one solution, then we are done. So, we only need to show that if the system has more than one solution, then it has infinitely many solutions. Assume that Y and Z are two solutions for the system $AX = B$. Then, for $r, s \in \mathbb{R}$, define $U = r\,Y + s\,Z$ to get:

$$AU = A(r\,Y + s\,Z) = A(r\,Y) + A(s\,Z) = r\,(AY) + s\,(AZ) = r\,B + s\,B = (r+s)B.$$

If $r + s = 1$, then U is a solution to the system. Since there are infinitely many choices for r and s in $\mathbb{R}$, the system has infinitely many solutions.

TRUE or FALSE:

$\star$ If X_1 and X_2 are two solutions for $AX = B$, then $\frac{3}{2}X_1 - 2X_2$ is also a solution. (FALSE).

reason: Since $\frac{3}{2} - 2 = \frac{-1}{2} \neq 1$.

Theorem 1.6.2

If A is a nonsingular $n \times n$ matrix, then for each $n \times 1$ matrix B, the system of equations $AX = B$ has a unique solution, namely $X = A^{-1}B$.

Proof:

Given $AX = B$, we multiply both sides (from left) by A^{-1} to get $X = A^{-1}B$. That is $A^{-1}B$ is a solution to the system.

To show that it is unique, assume that Y is any solution for $AX = B$. Hence $AY = B$ and again $Y = A^{-1}B$.

Example 1.6.1

Solve the following system "$A\,X = B$" (given in its matrix form)

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

Solution:

We solve the system by using A^{-1}, we can find the inverse of A as in Example 1.5.2, to get

$$A^{-1} = \begin{bmatrix} -1 & 0 & 1 \\ -3 & 1 & 1 \\ 2 & 0 & -1 \end{bmatrix}$$

Then, using Theorem 1.6.2, we get the following unique solution:

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1}\,B = \begin{bmatrix} -1 & 0 & 1 \\ -3 & 1 & 1 \\ 2 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

Example 1.6.2

Solve the following system "$A\,X = O$"

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Solution:

We can find the inverse of A as in Example 1.5.2, to get

$$A^{-1} = \begin{bmatrix} -1 & 0 & 1 \\ -3 & 1 & 1 \\ 2 & 0 & -1 \end{bmatrix}$$

Then, using Remark 1.6.2, we get the trivial solution:

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1}\,O = \begin{bmatrix} -1 & 0 & 1 \\ -3 & 1 & 1 \\ 2 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

• Solving a sequence of systems with a common coefficient matrix

Given the systems $AX = B_1; AX = B_2; \cdots ; AX = B_k$, an efficient way to solve these systems at once is to form the augmented matrix

$$[\ A \mid B_1 \mid B_2 \mid \cdots \mid B_k\ ],$$

and use the Guass-Jordan elimination to solve all of the system at the same time.

Example 1.6.3

Solve the following linear systems:

$$\begin{array}{rcrcrcl} x & & & + & z & = & b_1 \\ x & + & y & + & 2z & = & b_2 \\ 2x & & & + & 3z & = & b_3 \end{array} \qquad \text{for} \qquad \begin{array}{rl} \text{i.} & b_1 = 0,\ b_2 = 1, \text{ and } b_3 = 1; \\ \text{ii.} & b_1 = 1,\ b_2 = 2, \text{ and } b_3 = 3. \end{array}$$

Solution:

$$\left[\begin{array}{ccc|c|c} 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 2 & 1 & 2 \\ 2 & 0 & 3 & 1 & 3 \end{array}\right] \xrightarrow[r_3 - 2r_1 \to r_3]{r_2 - r_1 \to r_2} \left[\begin{array}{ccc|c|c} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{array}\right] \xrightarrow[r_2 - r_3 \to r_2]{r_1 - r_2 \to r_1} \left[\begin{array}{ccc|c|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{array}\right]$$

Therefore, the solution of the system (i) is $x = -1, y = 0, z = 1$ and the solution of system (ii) is $x = 0, y = 0, z = 1$.

Theorem 1.6.3

Let A be a square matrix. Then

1. If B is a square matrix satisfying $BA = I$, then $B = A^{-1}$.
2. If B is a square matrix satisfying $AB = I$, then $B = A^{-1}$.

Proof:

1. Assuming $BA = I$, we show that A is nonsingular by showing that the system $AX = O$ has only the trivial solution. Multiplying both sides (from left) by B, we get $B\,(AX) = B\,O$ which implies $(BA)X = IX = X = O$. Thus, the system $AX = O$ has only the

trivial solution. Theorem 1.5.1 implies that A is nonsingular. Therefore $BA = I$ implies $(BA)A^{-1} = IA^{-1}$. Hence $B = A^{-1}$.

2. Using part (a), $AB = I$ implies that $A = B^{-1}$. Taking the inverse for both sides, we get $A^{-1} = B$ as desired.

Theorem 1.6.4: Extended Version of Theorem 1.5.1

If A is an $n \times n$ matrix, then the following statements are equivalent:

1. A is nonsingular.
2. $AX = O$ has only the trivial solution.
3. A is row equivalent to I_n.
4. $AX = B$ is consistent for every $n \times 1$ matrix B.
5. $AX = B$ has exactly one solution for every $n \times 1$ matrix B.

Proof:

We only show that 5 implies 1: Assume that the system $AX = B$ has one solution for every $n \times 1$ matrix B. Let $X_1, X_2, \cdots, X_n$ be the solutions (respectively) to the following systems:

$$AX = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad AX = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \cdots, \quad AX = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}.$$

If C is the $n \times n$ matrix formed by the column vectors $X_1, X_2, \cdots, X_n$, we get

$$AC = A\,[X_1 \mid X_2 \mid \cdots \mid X_n] = [AX_1 \mid AX_2 \mid \cdots \mid AX_n] = I_n$$

By Theorem 1.6.3, we get $C = A^{-1}$. Thus A is nonsingular.

Theorem 1.6.5

Let A and B be two square matrices of the same size. If AB is nonsingular, then A and B must be also nonsingular.

In what follows, we discuss when a given system is consistent based on the remarks in Section 1.2

Example 1.6.4

Consider the system:

$$\begin{array}{ccccccc} x & + & y & + & z & = & 2 \\ x & + & 2y & - & z & = & 2 \\ x & + & 2y & + & (a^2-5)z & = & a \end{array}$$

Find all values of a so that the system has:

(a) a unique solution (consistent).
(b) infinite many solutions (consistent).
(c) no solution (inconsistent).

Solution:

In this kind of questions, it is not necessary to get the r.r.e.f. So, we will try to focus on the last row which contains the term "a" as follows:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 1 & 2 & -1 & 2 \\ 1 & 2 & a^2-5 & a \end{array}\right] \xrightarrow[r_3-r_1\to r_3]{r_2-r_1\to r_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & -2 & 0 \\ 0 & 1 & a^2-6 & a-2 \end{array}\right] \xrightarrow{r_3-r_2\to r_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & a^2-4 & a-2 \end{array}\right]$$

At this point, we have:

(a) a **unique solution** if $a^2-4\neq 0 \iff a\neq \pm 2 \iff a\in\mathbb{R}\backslash\{-2,+2\}$.
(b) **infinite solutions** if $\left\{a^2-4=0 \text{ and } a-2=0\right\} \iff \left\{a=\pm 2 \text{ and } a=+2\right\} \iff a=+2$.
(c) **no solution** if $\left\{a^2-4=0 \text{ and } a-2\neq 0\right\} \iff \left\{a=\pm 2 \text{ and } a\neq +2\right\} \iff a=-2$.

Example 1.6.5

Discuss the consistency of the following non-homogenous system:

$$\begin{array}{ccccccc} x & + & 3y & + & kz & = & 4 \\ 2x & + & ky & + & 12z & = & 6 \end{array}$$

Solution:

$$\left[\begin{array}{ccc|c} 1 & 3 & k & 4 \\ 2 & k & 12 & 6 \end{array}\right] \xrightarrow{r_2-2r_1\to r_2} \left[\begin{array}{ccc|c} 1 & 3 & k & 4 \\ 0 & k-6 & 12-2k & -2 \end{array}\right].$$

The system is consistent only if $k \neq 6$:

(a) can not have a unique solution (not equivalent to I because it is not square matrix!!),

(b) has infinite solutions when $k \neq 6$,

(c) has no solutions when $k = 6$.

Exercise 1.6.1

1. Let $A = \begin{bmatrix} 1 & 0 & 3 & 3 \\ 0 & 1 & 1 & -1 \\ 1 & -2 & 3 & 1 \\ 0 & 2 & 0 & a^2+1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 \\ 0 \\ 0 \\ a+2 \end{bmatrix}$. Find all value(s) of a such that the system $AX = B$ is consistent.

2. Find all value(s) of a for which the system
$$\begin{array}{rcrcrcl} x & - & y & + & (a+3)z & = & a^3 - a - 7 \\ -x & + & ay & - & az & = & a \\ & & 2(a-1)y & + & (a^2+2)z & = & 8a - 14. \end{array}$$
has (a) no solution (inconsistent), (b) unique solution (consistent), and (c) infinite many solutions (consistent).

3. Discuss the consistency of the following homogenous system:
$$\begin{array}{rcrcrcl} x & + & y & - & z & = & 0 \\ x & - & y & + & 3z & = & 0 \\ x & + & y & + & (a^2-5)z & = & 0 \end{array}$$

4. Show that if C_1 and C_2 are solutions of the system $AX = B$, then $4C_1 - 3C_2$ is also a solution of this system.

5. Let U and V be two solutions of the homogenous system $AX = O$. Show that $rU + sV$ (for $r, s \in \mathbb{R}$) is a solution to the same system.

6. Let U and V be two solutions of the non-homogenous system $AX = B$. Show that $U - V$ is a solution to the homogenous system $AX = O$.

7. If A is nonsingular matrix, then AA^T and A^TA are both nonsingular matrices.

8. Show that if A, B, and $A+B$ are invertible matrices with the same size, then
$$A\left(A^{-1} + B^{-1}\right) B\,(A+B)^{-1} = I.$$

9. Let A be an $m \times n$ matrix, and B be an $m \times 1$ column vector. Show that the system $AX = B$ has a solution if and only if B is a linear combination of columns of A. [Hint: Recall Definition 1.3.4].

1.7 Diagonal, Triangular, and Symmetric Matrices

The "**diagonal matrix**", $D = [d_{ij}]$, is an $n \times n$ matrix so that $d_{ij} = 0$ for all $i \neq j$.

$$D = \begin{bmatrix} d_{11} & 0 & \cdots & 0 \\ 0 & d_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_{nn} \end{bmatrix} \text{ it is also can be written as } D = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix}$$

Moreover, we sometime write $D = diag(d_1, d_2, \cdots, d_n)$. If all scalars in the diagonal matrix are equal, say equal c, then D is said to be a scalar matrix. In particular, the identity matrix I_n is a scalar matrix with $c = 1$. That is $I_n = diag(1, 1, \cdots, 1)$.

1. D is nonsingular if and only if all of its diagonal entries are nonzero; in this case we have

$$D^{-1} = \begin{bmatrix} 1/d_1 & 0 & \cdots & 0 \\ 0 & 1/d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1/d_n \end{bmatrix}$$

2. If k is a positive integer, then D^k is computed as

$$D^k = \begin{bmatrix} d_1^k & 0 & \cdots & 0 \\ 0 & d_2^k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n^k \end{bmatrix}$$

3. If A is an $n \times m$ matrix and B is an $m \times n$ matrix, then

$$D\,A = \begin{bmatrix} d_1\, row_1(A) \\ d_2\, row_2(A) \\ \vdots \\ d_n\, row_n(A) \end{bmatrix} \quad \text{and} \quad B\,D = \begin{bmatrix} d_1\, col_1(B) & d_2\, col_2(B) & \cdots & d_n\, col_n(B) \end{bmatrix}$$

Example 1.7.1

Let $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$, $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 0 \\ 2 & 1 & 4 \\ 5 & 1 & -1 \end{bmatrix}$, and $B = \begin{bmatrix} 1 & 1 & -1 & 0 \\ 2 & 3 & 1 & 2 \\ 4 & 2 & 1 & 3 \end{bmatrix}$. Compute D^{-3}, AD, and DB.

Solution:

Note that $D = diag(1, 3, 2)$. Hence $D^{-1} = diag\left(1, \frac{1}{3}, \frac{1}{2}\right)$ and thus $D^{-3} = \left(D^{-1}\right)^3 = diag\left(1, \frac{1}{27}, \frac{1}{8}\right)$.

$$AD = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 0 \\ 2 & 1 & 4 \\ 5 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -2 \\ 1 & 3 & 0 \\ 2 & 3 & 8 \\ 5 & 3 & -2 \end{bmatrix} = \begin{bmatrix} col_1(A) & 3\,col_2(A) & 2\,col_3(A) \end{bmatrix},$$

and

$$DB = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 & 0 \\ 2 & 3 & 1 & 2 \\ 4 & 2 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 & 0 \\ 6 & 9 & 3 & 6 \\ 8 & 4 & 2 & 6 \end{bmatrix} = \begin{bmatrix} row_1(B) \\ 3\,row_2(B) \\ 2\,row_3(B) \end{bmatrix}.$$

Example 1.7.2

Find all 2×2 diagonal matrices A that satisfy the equation $A^2 - 3A + 2I = O$.

Solution:

Assume that $A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ be a 2×2 diagonal matrix. Then,

$$A^2 - 3A + 2I = \begin{bmatrix} a^2 & 0 \\ 0 & b^2 \end{bmatrix} - \begin{bmatrix} 3a & 0 \\ 0 & 3b \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Hence, $a^2 - 3a + 2 = 0$ and $b^2 - 3b + 2 = 0$. That is $(a-1)(a-2) = 0$ and $(b-1)(b-2) = 0$ which implies that $a = 1$ or 2 and $b = 1$ or 2. Therefore,

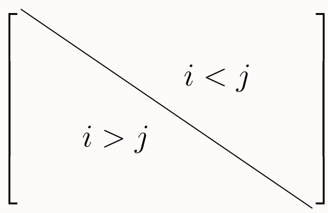
$$A \in \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \right\}.$$

The "**lower triangular matrix**", $L = [l_{ij}]$, is an $n \times n$ matrix so that $l_{ij} = 0$ for all $i < j$. The "**upper triangular matrix**", $U = [u_{ij}]$, is an $n \times n$ matrix so that $u_{ij} = 0$ for all $i > j$.

$$L = \begin{bmatrix} l_{11} & & & \\ l_{21} & l_{22} & & \text{\Large 0} \\ \vdots & \vdots & \ddots & \\ l_{n1} & l_{n2} & \cdots & l_{nn} \end{bmatrix}, \text{ and } \quad U = \begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ & u_{22} & \cdots & u_{2n} \\ \text{\Large 0} & & \ddots & \vdots \\ & & & u_{nn} \end{bmatrix}$$

Theorem 1.7.1

- The transpose of a lower triangular matrix is upper triangular matrix, and the transpose of an upper triangular matrix is lower triangular matrix.
- The product of lower triangular matrices is lower triangular, and the product of upper triangular matrices is upper triangular.
- A triangular matrix is nonsingular if and only if its diagonal entries are all nonzero.
- The inverse of a nonsingular lower triangular matrix is lower triangular, and the inverse of a nonsingular upper triangular matrix is upper triangular.



Example 1.7.3

Find a lower triangular matrix that satisfies $A^3 = \begin{bmatrix} 8 & 0 \\ 9 & -1 \end{bmatrix}$.

Solution:

Assume that $A = \begin{bmatrix} a & 0 \\ b & c \end{bmatrix}$ be a 2×2 lower triangular matrix. Then,

$$A^3 = \begin{bmatrix} a^3 & 0 \\ a^2b + c(ab + bc) & c^3 \end{bmatrix} = \begin{bmatrix} 8 & 0 \\ 9 & -1 \end{bmatrix}.$$

Hence, $a = 2$ and $c = -1$ and thus $4b - (2b - b) = 9$ implies that $b = 3$.

Definition 1.7.1

A square matrix is called **symmetric** if $A^T = A$. It is called **skew-symmetric** if $A^T = -A$.

A square matrix $A = [a_{ij}]$ is symmetric if $a_{ij} = a_{ji}$ and it is skew-symmetric if $a_{ij} = -a_{ji}$.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 0 \end{bmatrix} \text{ is a symmetric; where } B = \begin{bmatrix} 0 & 2 & -3 \\ -2 & 0 & 1 \\ 3 & -1 & 0 \end{bmatrix} \text{ is a skew-symmetric.}$$

Example 1.7.4

Fill in the missing entries (marked with $\times$) to produce symmetric (or skew-symmetric) matrices.

$$\begin{bmatrix} \times & 2 & \times \\ \times & \times & 0 \\ -2 & \times & \times \end{bmatrix}.$$

Theorem 1.7.2

If A and B are symmetric matrices with the same size, and k is any scalar, then:

1. A^T is symmetric.
2. $A + B$ and $A - B$ are symmetric.
3. kA is symmetric.
4. AB is symmetric iff $AB = BA$.

Proof:

Note that A and B are symmetric matrices and hence $A^T = A$ and $B^T = B$. Then,

1. $(A^T)^T = A = A^T$. Then A^T is symmetric.
2. $(A \pm B)^T = A^T \pm B^T = A \pm B$. Then $A + B$ and $A - B$ are symmetric.
3. $(kA)^T = kA^T = kA$. Then, kA is symmetric.
4. AB is symmetric iff $(AB)^T = AB$ iff $B^T A^T = AB$ iff $BA = AB$.

Theorem 1.7.3

If A is a symmetric nonsingular matrix, then A^{-1} is symmetric.

Proof:

Assume that A is symmetric nonsingular matrix. Then

$$(A^{-1})^T = A^{T^{-1}} = A^{-1}.$$

Therefore, A^{-1} is symmetric.

Example 1.7.5

Show that A^TA and AA^T are both symmetric matrices.

Solution:

It is clear that $(A^TA)^T = A^T(A^T)^T = A^TA$ which shows that A^TA is symmetric. In addition, $(AA^T)^T = (A^T)^TA^T = AA^T$ shows that AA^T is also symmetric.

Example 1.7.6

1. If A is nonsingular skew-symmetric matrix, then A^{-1} is skew-symmetric.
2. If A and B are skew-symmetric matrices, then so are A^T, $A+B$, $A-B$ and kA for any scalar k.
3. Every square matrix A can be expressed as the sum of a symmetric matrix and a skew-symmetric matrix.

Solution:

1. Assume that $A^T = -A$. Then $(A^{-1})^T = (A^T)^{-1} = (-A)^{-1} = -A^{-1}$. That is A^{-1} is skew-symmetric.
2. We only show that $A+B$ is skew-symmetric: $(A+B)^T = A^T + B^T = -A + (-B) = -(A+B)$.
3. If A is any square matrix, then $A = \frac{1}{2}\left(A + A^T\right) + \frac{1}{2}\left(A - A^T\right)$. Then we only need to show that $\frac{1}{2}\left(A + A^T\right)$ and $\frac{1}{2}\left(A - A^T\right)$ are symmetric and skew-symmetric matrices, respectively.

Exercise 1.7.1

1. Show that if A and B are symmetric matrices, then $AB - BA$ is a skew-symmetric matrix.
2. Let A be 2×2 skew-symmetric matrix. If $A^2 = A$, then $A = \mathbf{0}$.
3. If A and B are lower triangular matrices, show that $A + B$ is lower triangular as well.
4. If A and B are skew-symmetric matrices and $AB = BA$, then AB is symmetric.
5. Let $A \in M_{n\times n}$. Show that

   (a) $A^T + A$ is symmetirc.

   (b) $A - A^T$ is skew-symmetric.

6. Let $A \in M_{n\times n}$. Then A can be written as $A = S + K$, where S is symmetric matrix and K is skew-symmetric matrix.
7. Let $A = \begin{bmatrix} 2 & -1 & 3 \\ 0 & 4 & 1 \\ 1 & -2 & -3 \end{bmatrix}$. Find a symmetric matrix S and a skew symmetric matrix K such that $A = S + K$.

Chapter

2 Determinants

2.1 Determinants

Definition 2.1.1

Let $S = \{1, 2, \cdots, n\}$ be a set of integers from 1 to n. A **rearrangement** of $j_1, j_2, \cdots, j_n$ of elements of S is called **permutation** of S.

Example 2.1.1

For $S = \{1, 2\}$, we have 2 permutations:

$$1\,2, \text{ and } 2\,1.$$

While, for $S = \{1, 2, 3\}$, we have the following 6 permutations:

$$1\,2\,3,\ 1\,3\,2,\ 2\,1\,3,\ 2\,3\,1,\ 3\,1\,2, \text{ and } 3\,2\,1.$$

In general, for $S = \{1, 2, \cdots, n\}$, we have $n! = n(n-1)(n-2)\cdots 3 \cdot 2 \cdot 1$ permutations.

Definition 2.1.2

A permutation $j_1 j_2 \cdots j_n$ of the set $S = \{1, 2, \cdots, n\}$ is said to have an **inversion** if a larger integer j_r preceedes a smaller one j_s for $r, s \in S$. A permutation is called **even** with a **positive "+" sign** or **odd** with a **negative "−" sign** according to whether the total number of inversions in it is even or odd, respectively.

Example 2.1.2

If $S = \{1, 2, 3\}$, then

permutation	#inversions	even-odd	sign	inversions
1 2 3	0	even	$+$	1 2 1 3 2 3
1 3 2	1	odd	$-$	1 3 1 2 $\boxed{3\ 2}$
2 1 3	1	odd	$-$	$\boxed{2\ 1}$ 2 3 1 3
2 3 1	2	even	$+$	2 3 $\boxed{2\ 1}$ $\boxed{3\ 1}$
3 1 2	2	even	$+$	$\boxed{3\ 1}$ $\boxed{3\ 2}$ 1 2
3 2 1	3	odd	$-$	$\boxed{3\ 2}$ $\boxed{3\ 1}$ $\boxed{2\ 1}$

TRUE or FALSE:

$\star$ The permutation 5 2 1 3 4 has a positive sign. (FALSE).

reason: The number of inversions is 5 which are 5 2, 5 1, 5 3, 5 4, and 2 1. So, this permutation has an odd number of inversions and a negative sign.

Definition 2.1.3

Let $A = (a_{ij}) \in M_{n\times n}$. Then, the determinant of A, denoted by $\det(A)$ or $\left|A\right|$, is

$$\det(A) = \sum(\pm)a_{1j_1}a_{2j_2}a_{3j_3}\cdots a_{nj_n}$$

where the summation ranges over all permutations $j_1 j_2 \cdots j_n$ of the set $S = \{1, 2, \cdots, n\}$. The sign is taken as $+$ or $-$ according to the sign of the permutation.

Example 2.1.3

Compute the determinant of $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$.

Solution:

Using the definition, we have $\det(A) = \sum(\pm)a_{1j_1}a_{2j_2}$, where $j_1 j_2$ is a permutation of $S = \{1, 2\}$. Thus, $j_1 j_2 \in \{1\,2, 2\,1\}$ and

$$\det(A) = +\ a_{11}a_{22} -\ a_{12}a_{21}.$$

Example 2.1.4

Compute the determinant of $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$.

Solution:

Using the definition, we have $\det(A) = \sum(\pm)a_{1j_1}a_{2j_2}a_{3j_3}$, where $j_1j_2j_3$ is a permutation of $S = \{1, 2, 3\}$. Thus, $j_1j_2j_3 \in \{1\,2\,3, 1\,3\,2, 2\,1\,3, 2\,3\,1, 3\,1\,2, 3\,2\,1\}$ and

$$\det(A) = +\, a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}.$$

Moreover, this formula can be found by taking the sum of the positive product of the diagonal entries and the negative product of the anti-diagonal entries in the following matrix:

$$\begin{array}{ccccc} + & + & +- & - & - \\ a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{array}$$

Determinants of 2×2 and 3×3 matrices can be evaluated using

$$A_1 = \begin{array}{cc} + & - \\ a_{11} & a_{12} \\ a_{21} & a_{22} \end{array} \qquad A_2 = \begin{array}{ccccc} + & + & +- & - & - \\ a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{array}$$

For the 2×2 matrix A_1, we get $\det(A_1) = a_{11}a_{22} - a_{12}a_{21}$. It is simply the result of "blue stripe" product minus "red stripe" product.

While for the 3×3 matrix A_2, we first recopy the first two columns and then we add up the product of the blue stripes and subtract the product of the red stripes.

$$\det(A_2) = \big(a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}\big) - \big(a_{13}a_{22}a_{31} + a_{11}a_{23}a_{32} + a_{12}a_{21}a_{33}\big).$$

For example, $\begin{vmatrix} 3 & 2 \\ 4 & 2 \end{vmatrix} = -2$, and $\begin{vmatrix} 3 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & 1 \end{vmatrix} = 2$.

Definition 2.1.4

If A is a square matrix, then the **minor of entry** a_{ij} is denoted by M_{ij} and is defined to be the determinant of the submatrix that remains after the i^{th} row and j^{th} column are deleted from A. The number $(-1)^{i+j}\ M_{ij}$ is denoted by A_{ij} and is called the **cofactor of entry** a_{ij} . Moreover, the **cofactor matrix** denoted by $\text{coef}(A) = [A_{ij}]$ where $1 \le i, j \le n$.

Example 2.1.5

Compute the minors, the cofactors, and the cofactor matrix of A, where $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 3 & 1 & -2 \end{bmatrix}$.

Solution:

$$M_{11} = \begin{vmatrix} 1 & -1 \\ 1 & -2 \end{vmatrix} = -1, \qquad M_{12} = \begin{vmatrix} 0 & -1 \\ 3 & -2 \end{vmatrix} = 3, \qquad M_{13} = \begin{vmatrix} 0 & 1 \\ 3 & 1 \end{vmatrix} = -3,$$

$$M_{21} = \begin{vmatrix} 2 & 1 \\ 1 & -2 \end{vmatrix} = -5, \qquad M_{22} = \begin{vmatrix} 1 & 1 \\ 3 & -2 \end{vmatrix} = -5, \qquad M_{23} = \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} = -5,$$

$$M_{31} = \begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} = -3, \qquad M_{32} = \begin{vmatrix} 1 & 1 \\ 0 & -1 \end{vmatrix} = -1, \qquad M_{33} = \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} = 1.$$

Thus, the cofactors are:

$$A_{11} = (-1)^2 M_{11} = -1, \qquad A_{12} = (-1)^3 M_{11} = -3, \qquad A_{13} = (-1)^4 M_{11} = -3,$$

$$A_{21} = (-1)^3 M_{11} = 5, \qquad A_{22} = (-1)^4 M_{11} = -5, \qquad A_{23} = (-1)^5 M_{11} = 5,$$

$$A_{31} = (-1)^4 M_{11} = -3, \qquad A_{32} = (-1)^5 M_{11} = 1, \qquad A_{33} = (-1)^6 M_{11} = 1.$$

Therefore, $\text{coef}(A) = \begin{bmatrix} -1 & -3 & -3 \\ 5 & -5 & 5 \\ -3 & 1 & 1 \end{bmatrix}$.

We note that the minors M_{ij} and the cofactors A_{ij} are either the same or the negative of each other. This results from the $(-1)^{i+j}$ entry in the cofactor value.

If $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, then, its sign matrix is $\begin{bmatrix} + & - \\ - & + \end{bmatrix}$, and its minros and cofactors are:

$$A_{11} = M_{11} = a_{22}; \quad A_{12} = -M_{12} = -a_{21}; \quad A_{21} = -M_{21} = -a_{12}; \quad \text{and } A_{22} = M_{22} = a_{11}.$$

Definition 2.1.5

If A is an $n \times n$ matrix, then the number resulted by the sum of multiplying the entries in any row by the corresponding cofactors is called the **determinant of** A, and the sums themselves are called **cofactor expansion of** A.

That is, the determinant of A using the cofactor expansion along the i^{th} row is:

$$\det(A) = a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{in}A_{in}.$$

While the determinant of A using the cofactor expansion along the j^{th} column is:

$$\det(A) = a_{1j}A_{1j} + a_{2j}A_{2j} + \cdots + a_{nj}A_{nj}.$$

Theorem 2.1.1

Let A be an $n \times n$ matrix, then for each $1 \leq i \leq n$ we have

$$a_{i1}A_{k1} + a_{i2}A_{k2} + \cdots + a_{in}A_{kn} = \begin{cases} \det(A) & \text{if } i = k, \\ 0 & \text{if } i \neq k. \end{cases}$$

and for each $1 \leq j \leq n$

$$a_{1j}A_{1k} + a_{2j}A_{2k} + \cdots + a_{nj}A_{nk} = \begin{cases} \det(A) & \text{if } j = k, \\ 0 & \text{if } j \neq k. \end{cases}$$

Example 2.1.6

Compute the determinant of A by using the cofactor expansion method, where A is the matrix of Example 2.1.5:

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 3 & 1 & -2 \end{bmatrix}$$

Solution:

Here we can choose any row or column to compute the determinant. Using the cofactor expansion along the 1^{st} row, we get

$$\det(A) = a_{11}\ A_{11} + a_{12}\ A_{12} + a_{13}\ A_{13} = (1)(-1) + (2)(-3) + (1)(-3) = -10.$$

Choosing the first row of A and the cofactors of (for instance) the second row of A, we get

$$a_{11}\ A_{21} + a_{12}\ A_{22} + a_{13}\ A_{23} = (1)(5) + (2)(-5) + (1)(5) = 0.$$

Example 2.1.7

Compute $\det(A)$ using the cofactor expansion method, where $A = \begin{bmatrix} 0 & 3 & 0 & 1 \\ 2 & 1 & 1 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \end{bmatrix}$.

Solution:

We use the cofactor expansion along the 1^{st}-column since it has the most zeros:

$$\begin{vmatrix} 0 & 3 & 0 & 1 \\ 2 & 1 & 1 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \end{vmatrix} = (0)\ A_{11} + (-1)^{2+1}(2) \begin{vmatrix} 3 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & 1 \end{vmatrix} + (0)\ A_{31} + (0)\ A_{41}$$

$$= (-2) \begin{vmatrix} 3 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & 1 \end{vmatrix} = (-2) \begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix} = (-2)\ [3 - 1] = -4.$$

Theorem 2.1.2

If A is an $n \times n$ triangular matrix (upper triangular, lower triangular, or diagonal), then

$$\det(A) = a_{11}\, a_{22}\, \cdots\, a_{nn}.$$

Example 2.1.8

We use Theorem 2.1.2 to compute:

$$\begin{vmatrix} 3 & 0 & 0 \\ 2 & 5 & 0 \\ 100 & 1987 & 2 \end{vmatrix} = 30.$$

Can you check the answer using another method?

Exercise 2.1.1

1. Let $D = \begin{bmatrix} 0 & 4 & 4 & 0 \\ 1 & 1 & 2 & 0 \\ 1 & 3 & 5 & 3 \\ 0 & 1 & 2 & 6 \end{bmatrix}$. Evaluate $\left|D\right|$.
   Final answer: $|D| = -12$.
2. Let $A = \begin{bmatrix} a+e & b+f \\ c & d \end{bmatrix}$, $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, and $C = \begin{bmatrix} e & f \\ c & d \end{bmatrix}$. Show that $\left|A\right| = \left|B\right| + \left|C\right|$.
3. Let $A = \begin{bmatrix} 1 & 4 & 2 \\ 5 & -3 & 6 \\ 2 & 3 & 2 \end{bmatrix}$. Compute the cofactors A_{11}, A_{12}, and A_{13}, and show that $5A_{11} - 3A_{12} + 6A_{13} = 0$.

2.2 Evaluating Determinants by Row Reduction

In this section, we introduce some basic properties and theorems to compute determinants.

Theorem 2.2.1

Let A be an $n \times n$ matrix.

1. If A is a square matrix with a row of zeros (or a column of zeros), then $\det(A) = 0$.
2. If A is a square matrix, then $\det(A) = \det(A^T)$.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{vmatrix}$$

3. If B is obtained from A by multiplying a single row (or a single column) by a scalar k, then $\det(B) = k\det(A)$. This result can be generalized as $\det(k\,A) = k^n \det(A)$.

$$\begin{vmatrix} k\,a_{11} & k\,a_{12} & k\,a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = k \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

4. If B is obtained from A by interchaning two rows (or columns), then $\det(B) = -\det(A)$.

$$\begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = - \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

5. If B is obtained from A by adding a multiple of one row (one column) to another row (column, respectively), then $\det(B) = \det(A)$.

$$\begin{vmatrix} a_{11} + k\,a_{21} & a_{12} + k\,a_{22} & a_{13} + k\,a_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

6. If A has two proportional rows (or columns), then $\det(A) = 0$.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ k\,a_{11} & k\,a_{12} & k\,a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = 0$$

7. If A and B are two square matrices, then $\det(AB) = \det(A)\det(B)$.

Example 2.2.1

Evaluate the determinant of the matrix A, where

$$A = \begin{bmatrix} 2 & -2 & 5 & 3 \\ 6 & -6 & 15 & 9 \\ 0 & 3 & 1 & 9 \\ 1 & 4 & 2 & -1 \end{bmatrix}.$$

Solution:

Note that the second row of A is 3 times the first one. Then, $\det(A) = 0$.

$$\begin{vmatrix} 2 & -2 & 5 & 3 \\ 6 & -6 & 15 & 9 \\ 0 & 3 & 1 & 9 \\ 1 & 4 & 2 & -1 \end{vmatrix} \overset{r_2-3r_1\to r_2}{=\!=} \begin{vmatrix} 2 & -2 & 5 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 3 & 1 & 9 \\ 1 & 4 & 2 & -1 \end{vmatrix} = 0.$$

Example 2.2.2

Evaluate the determinant of the matrix A, where

$$A = \begin{bmatrix} 2 & -2 & 5 \\ 0 & 3 & 1 \\ 1 & 4 & 2 \end{bmatrix}.$$

Solution:

Note that the second row of A is 3 times the first one. Then, $\det(A) = 0$.

$$\begin{aligned}
\begin{vmatrix} 2 & -2 & 5 \\ 0 & 3 & 1 \\ 1 & 4 & 2 \end{vmatrix} &\overset{r_1\Leftrightarrow r_3}{=\!=} (-1)\begin{vmatrix} 1 & 4 & 2 \\ 0 & 3 & 1 \\ 2 & -2 & 5 \end{vmatrix} \overset{r_3-2r_1\to r_3}{=\!=} (-1)\begin{vmatrix} 1 & 4 & 2 \\ 0 & 3 & 1 \\ 0 & -10 & 1 \end{vmatrix} \\
&\overset{c_2\Leftrightarrow c_3}{=\!=} (-1)(-1)\begin{vmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \\ 0 & 1 & -10 \end{vmatrix} \overset{r_3-r_2\to r_3}{=\!=} \begin{vmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & -13 \end{vmatrix} \\
&= (1)(1)(-13) = -13
\end{aligned}$$

Example 2.2.3

Evaluate the determinant of the matrix A, where

$$A = \begin{bmatrix} 2 & -2 & 5 & 1 \\ 3 & 3 & 1 & 3 \\ 1 & 1 & 2 & 1 \\ 4 & 3 & 5 & 6 \end{bmatrix}.$$

Solution:

Note that the second row of A is 3 times the first one. Then, $\det(A) = 0$.

$$\begin{vmatrix} 2 & -2 & 5 & 1 \\ 3 & 3 & 1 & 3 \\ 1 & 1 & 2 & 1 \\ 4 & 3 & 5 & 6 \end{vmatrix} = \begin{vmatrix} 0 & -4 & 1 & -1 \\ 0 & 0 & -5 & 0 \\ 1 & 1 & 2 & 1 \\ 0 & -1 & -3 & 2 \end{vmatrix} = \begin{vmatrix} -4 & 1 & -1 \\ 0 & -5 & 0 \\ -1 & -3 & 2 \end{vmatrix}$$

$$= (-5)\begin{vmatrix} -4 & -1 \\ -1 & 2 \end{vmatrix} = (-5)[-8-1] = 45.$$

Example 2.2.4

Let $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = -6$. Compute $\begin{vmatrix} 3g & 3h & 3i \\ 2a+d & 2b+e & 2c+f \\ d & e & f \end{vmatrix}$.

Solution:

We use the properties stated in Theorem 2.2.1:

$$\begin{vmatrix} 3g & 3h & 3i \\ 2a+d & 2b+e & 2c+f \\ d & e & f \end{vmatrix} \xrightarrow[r_2 \Leftrightarrow r_3]{r_1 \Leftrightarrow r_2} (-1)(-1)\begin{vmatrix} 2a+d & 2b+e & 2c+f \\ d & e & f \\ 3g & 3h & 3i \end{vmatrix} \xrightarrow[\frac{1}{3}r_3]{r_1 - r_2 \to r_1}$$

$$(3)\begin{vmatrix} 2a & 2b & 2c \\ d & e & f \\ g & h & i \end{vmatrix} \xrightarrow{\frac{1}{2}r_1} (3)(2)\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = (6)(-6) = -36.$$

Theorem 2.2.2

If A is an $n \times n$ non-singular matrix, then $\left|A\right| \neq 0$ and $\left|A^{-1}\right| = \dfrac{1}{|A|}$. This statement suggests that if $\left|A\right| = 0$, then A is singular matrix.

Proof:

Since A is non-singular, A^{-1} exists, then

$$\begin{aligned} AA^{-1} &= I_n && \text{(take the determinant for both sides)} \\ \left|A\,A^{-1}\right| &= \left|I_n\right| \\ \left|A\right|\left|A^{-1}\right| &= 1. \end{aligned}$$

Thus, $\left|A\right| \neq 0$ and $\left|A^{-1}\right| = \dfrac{1}{|A|}$.

Example 2.2.5

Show that if $A \in M_{n\times n}$ is skew-symmetric matrix and n is odd, then $\left|A\right| = 0$.

Solution:

Since A is skew-symmetric, then $A^T = -A$ and taking the determinant for both sides

$$\begin{aligned} \left|A^T\right| &= \left|-A\right| \\ \left|A^T\right| &= (-1)^n\left|A\right|, && \text{where } n \text{ is odd and } (-1)^n = -1. \\ \left|A\right| = \left|A^T\right| &= -\left|A\right|. \end{aligned}$$

Therefore, $\left|A\right| = -\left|A\right|$ which means that $\left|A\right| = 0$.

<u>TRUE or FALSE</u>:

⋆ If $A, B \in M_{n\times n}$ with $\left|A\right| = \left|B\right|$, then $A = B$. (FALSE).

reason: $I_2 \neq -I_2$ while $\left|I_2\right| = 1$ and $\left|-I_2\right| = (-1)^2\left|I_2\right| = 1$.

Example 2.2.6

Show that if $A^{-1} = A^T$, then $\left|A\right| = 1$ or $\left|A\right| = -1$.

Solution:

Since $A^{-1} = A^T$, we have

$$|A^{-1}| = |A^T| \Longleftrightarrow \frac{1}{|A|} = \left|A\right| \Longleftrightarrow \left|A\right|^2 = 1 \Longleftrightarrow \left|A\right| = \pm 1.$$

Example 2.2.7

Let $A, B \in M_{3\times 3}$ with $\left|A\right| = 2$ and $\left|B\right| = -2$. Find $\left|2\,A^{-1}\,(B^2)^T\right|$.

Solution:

$$\begin{aligned} \left|2\,A^{-1}\,(B^2)^T\right| &= 2^3 \left|A^{-1}\right|\left|B^2\right| = 8\,\frac{1}{|A|}\left|B\right|\left|B\right| \\ &= 8\,(\frac{1}{2})\,(-2)(-2) = 16. \end{aligned}$$

Exercise 2.2.1

1. Compute $\det(A)$ where $A = \begin{bmatrix} 1 & 0 & 0 & 3 \\ 2 & 7 & 0 & 6 \\ 0 & 6 & 3 & 0 \\ 2 & 3 & 1 & 5 \end{bmatrix}$.

   [Hint: Simply reduce A to a lower triangular matrix! Find a relation between the first and the fourth columns.]

2. Let $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = -6$. Compute $\begin{vmatrix} -a & -b & -c \\ 2d & 2e & 2f \\ 3g & 3h & 3i \end{vmatrix}$.

   Final answer: 36.

3. Solve for x:

$$\begin{vmatrix} x & 1 \\ 1 & x-1 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 0 \\ x & 3 & -2 \\ 1 & 5 & -1 \end{vmatrix}.$$

   Final answer: $x = 1$.

4. Given that $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 7$, evaluate $\begin{vmatrix} b_1 & b_2 & b_1 - 3b_3 \\ a_1 & a_2 & a_1 - 3a_3 \\ c_1 & c_2 & c_1 - 3c_3 \end{vmatrix}$.

   Final answer: 21.

5. Let $A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$, and $B = \begin{bmatrix} 2a_3 & 2a_2 & 2a_1 \\ b_3 - a_3 & b_2 - a_2 & b_1 - a_1 \\ c_3 + 3b_3 & c_2 + 3b_2 & c_1 + 3b_1 \end{bmatrix}$. If $\left|A\right| = -4$, find $\left|B\right|$.

   Final answer: 8.

6. Let $A, B \in M_{n\times n}$. Show that, if $AB = I_n$, then $BA = I_n$.
7. If $|AB| = 0$, then either $\left|A\right| = 0$ or $\left|B\right| = 0$.
8. If $AB = I_n$, then $\left|A\right| \neq 0$ and $\left|B\right| \neq 0$.
9. Show that if A is non-singular and $A^2 = A$, then $\left|A\right| = 1$.
10. Show that for any $A, B, C \in M_{n\times n}$, if $AB = AC$ and $\left|A\right| \neq 0$, then $B = C$.
11. Find all values of α for which the matrix $\begin{bmatrix} 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & \alpha & \alpha^2 \end{bmatrix}$ is singular.
12. Let A and B be two $n \times n$ matrices such that A is invertible and B is singular. Prove that $A^{-1}B$ is singular.

13. If A and B are 2×2 matrices with $\det(A) = 2$ and $\det(B) = 5$, compute $\left|3\,A^2(AB^{-1})^T\right|$.

14. Let A be a matrix with $A^{-1} = \begin{bmatrix} 7 & 1 & 0 & 3 \\ 2 & 0 & 0 & 0 \\ 1 & 3 & 5 & 4 \\ 6 & 2 & 0 & 5 \end{bmatrix}$. Find $\det(A)$.

Chapter

3 Euclidean Vector Spaces

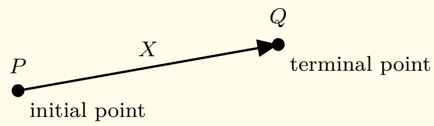
3.1 Vectors in $\mathbb{R}^n$

In this chapter, we deal with vectors in $\mathbb{R}^n$, or sometimes we simply call them n-vectors. For instance,

$$\text{(row-vector) } X = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}, \text{ and (column-vector) } Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \text{ are vectors in } \mathbb{R}^n.$$

In the notion of $\mathbb{R}^n$, A vector is simply written as $X = (x_1, x_2, \cdots, x_n)$. While A **points** is written as $X(x_1, x_2, \cdots, x_n)$.

Vecors in $\mathbb{R}^3$ can be manipulated by arrows starting at **initial point** and pointing at **terminal point**. In the Figure, the vector X has its initial point at P and has its terminal point at Q.



For instance, if $P(p_1, p_2, p_3)$ and $Q(q_1, q_2, q_3)$ are points in $\mathbb{R}^3$, then the vector X is written as $X = \overrightarrow{PQ} = (p_1 - q_1, p_2 - q_2, p_3 - q_3)$. This vector is different from its opposite vector $-X$ which is given by $-X = \overrightarrow{QP}$.

3.1.1 Vectors Operations:

Let $X = (x_1, x_2, \cdots, x_n), Y = (y_1, y_2, \cdots, y_n) \in \mathbb{R}^n$ and $c \in \mathbb{R}$. Then:

1. Adding or substracting two vectors, they must have the same number of components:

$$X \pm Y = (x_1 \pm y_1, x_2 \pm y_2, \cdots, x_n \pm y_n) \in \mathbb{R}^n.$$

2. Multiplying a vector by a scalar:

$$c\,X = (c\,x_1, c\,x_2, \cdots, c\,x_n) \in \mathbb{R}^n.$$

3. The vector $O = (0, 0, \cdots, 0) \in \mathbb{R}^n$ is called the **zero vector**.
4. Two vectors X and Y are said to be equal if $x_1 = y_1, x_2 = y_2, \cdots, x_n = y_n$.

Example 3.1.1

If $X = (2, 3, -1)$ and $Y = (1, 0, 1)$ are two vectors of $\mathbb{R}^3$, then $-X = (-2, -3, 1)$ and $3Y - X = (3, 0, 3) - (2, 3, -1) = (1, -3, 4)$.

Try to prove the statements in the following two Theorems.

Theorem 3.1.1: The Closure of $\mathbb{R}^n$ under Vector Addition and Scalar Multiplication

Let $X, Y, Z \in \mathbb{R}^n$ and let $c, d \in \mathbb{R}$. Then

1. $\mathbb{R}^n$ is **closed under vector addition** (i.e. $X + Y \in \mathbb{R}^n$):
   (a) $X + Y = Y + X$,
   (b) $X + (Y + Z) = (X + Y) + Z$,
   (c) $\exists$ a vector $O \in \mathbb{R}^n$ such that $X + O = O + X = X$. "additive identity"
   (d) for any $X \in \mathbb{R}^n$, $\exists (-X)$ such that $X + (-X) = (-X) + X = O$. "additive inverse"
2. $\mathbb{R}^n$ is **closed under scalar multiplication** (i.e. $c\,X \in \mathbb{R}^n$):
   (a) $c\,(X + Y) = c\,X + c\,Y$,
   (b) $(c + d)X = c\,X + d\,X$,
   (c) $c\,(d\,X) = (c\,d)\,X$,
   (d) $1\,X = X$, where $1 \in \mathbb{R}$.

Theorem 3.1.2

If X is a vector in $\mathbb{R}^n$ and c is any scalar, then

1. $0\,X = O$.
2. $c\,O = O$.
3. $(-1)\,X = -X$.

Definition 3.1.1

If $X \in \mathbb{R}^n$, then we say that X is a **linear combination** of the vectors $V_1, V_2, \cdots, V_n \in \mathbb{R}^n$, if it can be expressed as

$$X = c_1 V_1 + c_2 V_2 + \cdots + c_n V_n,$$

where $c_1, c_2, \cdots, c_n$ are scalars in $\mathbb{R}$. These scalars are called the **coefficients** of the linear combination.

Exercise 3.1.1

Find all scalars $c_1, c_2, c_3 \in \mathbb{R}$ such that

$$c_1(1, 2, 0) + c_2(2, 1, 1) + c_3(0, 0, 1) = (0, 0, 0).$$

Final answer: $c_1 = c_2 = c_3 = 0$. "Try to create a system of linear equations."

3.2 Norm and Dot Product in $\mathbb{R}^n$

Definition 3.2.1

Let $X = (x_1, x_2, \cdots, x_n)$ and $Y = (y_1, y_2, \cdots, y_n)$ in $\mathbb{R}^n$. Then

1. The **dot product** of X and Y is defined as

$$X \cdot Y = \sum_{i=1}^{n} x_i\, y_i = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n.$$

2. The **norm** (or **length**) of the vector X is defined as

$$\| X \| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}.$$

Remark 3.2.1

We note that, $\| X \|^2 = x_1^2 + x_2^2 + \cdots + x_n^2 = X \cdot X$. Therefore, $\| X \|^2 = X \cdot X$.

Theorem 3.2.1

If $X \in \mathbb{R}^n$ and $c \in \mathbb{R}$, then:

1. $\| X \| \geq 0$
2. $\| X \| = 0$ if and only if $X = O$.
3. $\| c\, X \| = |\, c \,|\, \| X \|$.

Proof:

The proof of the first two parts is easy. So we only prove the third statement of the Teorem. Let $X = (x_1, x_2, \cdots, x_n)$. Then $c\, X = (c\, x_1, c\, x_2, \cdots, c\, x_n)$. Thus,

$$\begin{aligned} \| c\, X \| &= \sqrt{(c\, x_1)^2 + (c\, x_2)^2 + \cdots + (c x_n)^2} = \sqrt{c^2\, (x_1^2 + x_2^2 + \cdots + x_n^2)} \\ &= |\, c \,| \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2} = |\, c \,|\, \| X \|. \end{aligned}$$

Definition 3.2.2

A vector of norm 1 is called a **unit** vector in $\mathbb{R}^n$. That is, U is a unit if $\| U \| = 1$.

Remark 3.2.2: Normalizing a Vector

If nonzero vector $X \in \mathbb{R}^n$, then $U = \dfrac{1}{\|X\|}X$ is the unit vector in the same direction as X.

$$\text{Clearly,} \qquad \| U \| = \left\| \frac{1}{\| X \|} X \right\| = \left| \frac{1}{\| X \|} \right| \| X \| = \frac{1}{\| X \|} \| X \| = 1.$$

The vectors $\mathbf{i} = (1, 0)$ and $\mathbf{j} = (0, 1)$ are called the **standard units** in $\mathbb{R}^2$.

The vectors $\mathbf{i} = (1, 0, 0)$, $\mathbf{j} = (0, 1, 0)$ and $\mathbf{k} = (0, 0, 1)$ are called the **standard units** in $\mathbb{R}^3$.

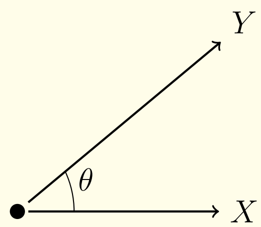
In general, the vectors $\mathbf{E}_1 = (1, 0, \cdots, 0), \mathbf{E}_2 = (0, 1, 0, \cdots, 0), \cdots, \mathbf{E}_n = (0, \cdots, 0, 1)$ are called the standard unit vectors in $\mathbb{R}^n$. Note that any vector $X = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$ is a linear combination of these vectors:

$$X = x_1\mathbf{E}_1 + x_2\mathbf{E}_2 + \cdots + x_n\mathbf{E}_n.$$

Remark 3.2.3: Another definition of dot product

If X and Y are nonzero vectors in $\mathbb{R}^n$ and if θ is the angle between X and Y, then

$$\cos\theta = \frac{X \cdot Y}{\| X \| \| Y \|} \qquad \text{where } 0 \le \theta \le \pi.$$



That is $X \cdot Y = \| X \| \| Y \| \cos\theta$.

Since, $-1 \le \cos\theta \le 1$, we get

$$-1 \le \frac{X \cdot Y}{\| X \| \| Y \|} \le 1.$$

Theorem 3.2.2

If $X, Y, Z \in \mathbb{R}^n$ and $c \in \mathbb{R}$. Then:

1. $O \cdot X = X \cdot O = 0$.

2. $X \cdot Y = Y \cdot X$.
3. $X \cdot (Y + Z) = X \cdot Y + X \cdot Z$ and $X \cdot (Y - Z) = X \cdot Y - X \cdot Z$.
4. $(X + Y) \cdot Z = X \cdot Z + Y \cdot Z$ and $(X - Y) \cdot Z = X \cdot Z - Y \cdot Z$
5. $(c\,X) \cdot Y = X \cdot (c\,Y) = c\,(X \cdot Y)$.
6. $X \cdot X \geq 0$ and $X \cdot X = \mathbf{0}$ if and only if $X = \mathbf{0}$.

Proof:

All parts are easy to prove.

Example 3.2.1

Find the angle between $X = (0, 1, 1, 0)$ and $Y = (1, 1, 0, 0)$.

Solution:

We have $\cos\theta = \dfrac{X \cdot Y}{\|\,X\,\|\,\|\,Y\,\|}$ where $X \cdot Y = 0 + 1 + 0 + 0 = 1$ and

$$\|\,X\,\| = \sqrt{0^2 + 1^2 + 1^2 + 0^2} = \sqrt{2} = \|\,Y\,\|.$$

Therefore, $\cos\theta = \dfrac{1}{2}$ which implies that $\theta = \dfrac{\pi}{3}$.

Theorem 3.2.3: Cauchy-Schwarz Inequality

If X and Y are vectors in $\mathbb{R}^n$, then $|\,X \cdot Y\,| \leq \|\,X\,\|\,\|\,Y\,\|$.

Theorem 3.2.4: Triangle Inequality

If $X, Y, Z \in \mathbb{R}^n$, then: $\|\,X + Y\,\| \leq \|\,X\,\| + \|\,Y\,\|$.

Proof:

By Remark 3.2.1, we have

$$\begin{aligned}
\|\,X + Y\,\|^2 &= (X + Y) \cdot (X + Y) = X \cdot X + X \cdot Y + Y \cdot X + Y \cdot Y \\
&= \|\,X\,\|^2 + 2\,X \cdot Y + \|\,Y\,\|^2 \qquad \text{absolute value.}
\end{aligned}$$

$$\begin{aligned} &= \|X\|^2 + 2\,|X \cdot Y| + \|Y\|^2 \qquad \text{Cauchy-Schwarz Inequality.} \\ &\leq \|X\|^2 + 2\,\|X\|\,\|Y\| + \|Y\|^2 = (\|X\| + \|Y\|)^2 . \end{aligned}$$

Example 3.2.2

If X and Y are vectors in $\mathbb{R}^n$, then $\|X+Y\|^2 + \|X-Y\|^2 = 2\left(\|X\|^2 + \|Y\|^2\right)$.

Solution:

$$\begin{aligned} \|X+Y\|^2 + \|X-Y\|^2 &= (X+Y)\cdot(X+Y) + (X-Y)\cdot(X-Y) \\ &= 2(X\cdot X) + 2X\cdot Y - 2X\cdot Y + 2(Y\cdot Y) \\ &= 2\Big(\|X\|^2 + \|Y\|^2\Big). \end{aligned}$$

Remark 3.2.4

If X and Y are in $\mathbb{R}^n$, then $\|X-Y\| \geq \Big|\,\|X\| - \|Y\|\,\Big|$.

Proof:

Recall that for real values x and a, we have $|x| \leq a$ iff $-a \leq x \leq a$. That is $a \geq x$ and $a \geq -x$. Therefore, we simply show that $\|X-Y\| \geq \|X\| - \|Y\|$ and $\|X-Y\| \geq \|Y\| - \|X\|$. First

$$\|X\| = \|(X-Y)+Y\| \leq \|X-Y\| + \|Y\| \;\rightarrow\; \|X\| - \|Y\| \leq \|X-Y\|.$$

For the second inequality, we use the first one (interchinging X and Y) in the following way:

$$\|X-Y\| = \|Y-X\| \geq \|Y\| - \|X\| \qquad \text{by the first inequality.}$$

Example 3.2.3

If $\|X\| = 2$ and $\|Y\| = 3$, what are the largest and smallest values possible for $\|X-Y\|$?

Solution:

By Triangle inequality, we have $\| X - Y \| \leq \| X \| + \| Y \| = 5$ which is the largest values of $\| X - Y \|$. For the smallest value, we use Remark 3.2.4. That is,

$$\| X - Y \| \geq \Big| \| X \| - \| Y \| \Big| = |2 - 3| = 1.$$

Exercise 3.2.1

1. Use Cauchy-SchwarzâĂŹs inequality to show that
$$(ab - cd + xy)^2 \leq (a^2 + d^2 + y^2)(b^2 + c^2 + x^2)$$
for all real numbers a, b, c, d, x, and y.
Hint: Find two suitable vectors in $\mathbb{R}^3$.
2. Let $U, V \in \mathbb{R}^n$ be unit vectors. Prove that $(U + 2V) \cdot (2U - V) \leq 3$.
3. Let θ be the angle between the vectors $U = (4, -2, 1, 2)$ and $V = (4, 2, 5, 2)$. Find $\cos\theta$.
Final answer: $\frac{21}{35}$.
4. For any vectors X and Y in $\mathbb{R}^n$, show that $\| X \| \leq \| X - 2Y \| + 2\| Y \|$.
Hint: Triangle inequality.
5. Let X and Y be two vectors in $\mathbb{R}^n$. Prove that $\|X - Y\| \leq \|X\| + \|Y\|$.
Hint: Triangle inequality.
6. Find a vector X, of length 6, in the opposite direction of $Y = (1, 2, -2)$.
Hint: What is $-6\,\frac{1}{\| Y \|}\,Y$?
7. Let X and Y be vectors in $\mathbb{R}^n$ such that $\|X\| = \|Y\|$. Show that $(X + Y) \cdot (X - Y) = 0$.
8. Let U and V be two vectors in $\mathbb{R}^3$ such that $\| U \| = 2$ and $\| V \| = 3$.
   (a) Find the maximum possible value for $\| 2U + 3V \|$.
   (b) If $U \cdot V = 0$, find $\| 2U + 3V \|$.
9. Let $X, Y \in \mathbb{R}^n$. Find $X \cdot Y$ given that $\| X + Y \| = 1$ and $\| X - Y \| = 5$.
Final answer: -6.
10. Answer each of the following as True or False:
   (a) If U and V are two unit vectors in $\mathbb{R}^n$, then $\| U - 6V \| \geq 5$.
   (b) There exist $X, Y \in \mathbb{R}^4$ such that $\| X \| = \| Y \| = 2$ and $X \cdot Y = 6$.
11. Find all values of a for which $X \cdot Y = 0$, where $X = (a^2 - a, -3, -1)$ and $Y = (2, a - 1, 2a)$.
Final answer: $a = \frac{1}{2}$ or 3.
12. If X and Y are vectors in $\mathbb{R}^n$, then $X \cdot Y = \dfrac{1}{4}\| X + Y \|^2 - \dfrac{1}{4}\| X - Y \|^2$.
13. Show that if $X \cdot Y = 0$ for all $Y \in \mathbb{R}^n$, then $X = O$. Use the standard unit vectors of $\mathbb{R}^n$ for Y.

14. Show that if $X \cdot Z = Y \cdot Z$ for all $Z \in \mathbb{R}^n$, then $X = Y$. Use the standard unit vectors of $\mathbb{R}^n$ for Z.

Chapter

4 General Vector Spaces

4.1 Real Vector Spaces

Definition 4.1.1

A **real vector space** $\mathbb{V}$ is a set of elements with two operations $\oplus$ and $\odot$ satisfying the following conditions. For short, we write $(\mathbb{V}, \oplus, \odot)$ is a vector space if

(α) if $X, Y \in \mathbb{V}$, then $X \oplus Y \in \mathbb{V}$, that is "$\mathbb{V}$ is closed under $\oplus$": for all $X, Y, Z \in \mathbb{V}$

(a) $X \oplus Y = Y \oplus X$,

(b) $X \oplus (Y \oplus Z) = (X \oplus Y) \oplus Z$,

(c) there exists $O \in \mathbb{V}$ such that $X \oplus O = O \oplus X = X$,

(d) for each $X \in \mathbb{V}$, there exists $e \in \mathbb{V}$ such that $e \oplus X = X \oplus e = O$.

(β) if $X \in \mathbb{V}$ and $c \in \mathbb{R}$, then $c \odot X \in \mathbb{V}$, that is "$\mathbb{V}$ is closed under $\odot$": for all $X, Y \in \mathbb{V}$ and for all $c, d \in \mathbb{R}$

(a) $c \odot (X \oplus Y) = c \odot X \oplus c \odot Y$,

(b) $(c + d) \odot X = c \odot X \oplus d \odot X$,

(c) $c \odot (d \odot X) = (c\, d) \odot X$,

(d) $1 \odot X = X \odot 1 = X$.

Remark 4.1.1

$(\mathbb{R}^n, +, \cdot)$ is a vector space. That is, $\mathbb{R}^n$ with vector addition and scalar multiplication is a vector space.

Example 4.1.1

Consider $\mathbb{V} = \{(x, y, z) : x, y, z \in \mathbb{R}\}$ with

$$(x_1, y_1, z_1) \oplus (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2),$$

and

$$c \odot (x, y, z) = (cx, cy, 0).$$

Is $(\mathbb{V}, \oplus, \odot)$ a vector space? Explain.

Solution:

Clearly, the α conditions are satisfied because this is the usual vector addition and hence $\mathbb{V}$ is closed under $\oplus$. Thus, we only check on the β conditions. Let $X = (x_1, y_1, z_1)$, $Y = (x_2, y_2, z_2)$ be any two vectors in $\mathbb{V}$, then

1. $c \odot (X + Y) = c \odot (x_1 + x_2, y_1 + y_2, z_1 + z_2) = (cx_1 + cx_2, cy_1 + cy_2, 0) = (cx_1, cy_1, 0) + (cx_2, cy_2, 0) = c \odot X + c \odot Y$. This condition is satisfied.

2. $(c + d) \odot X = ((c + d)x_1, (c + d)y_1, 0) = (cx_1, cy_1, 0) + (dx_1, dy_1, 0) = c \odot X + d \odot X$. This condition is satisfied.

3. $c \odot (d \odot X) = c \odot (dx_1, dy_1, 0) = (cdx_1, cdy_1, 0) = (cd) \odot X$. This condition is satisfied.

4. $1 \odot X = (x_1, y_1, 0) \neq (x_1, y_1, z_1)$. This condition is **NOT** satisfied.

Therefore, $(\mathbb{V}, \oplus, \odot)$ is not a vector space.

Example 4.1.2

Let $\mathbb{V} = \{(x, y, z) : x, y, z \in \mathbb{R} \text{ and } z > 0\}$ associated with the operations:

$$(x_1, y_1, z_1) \oplus (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2), \quad \text{and } c \odot (x, y, z) = (cx, cy, cz).$$

Is $(\mathbb{V}, \oplus, \odot)$ a vector space? Explain.

Solution:

No. If $c \in \mathbb{R}$ with $c < 0$, then $c \odot (x, y, z) = (cx, cy, cz) \notin \mathbb{V}$ since $cz < 0$.

Example 4.1.3

Is the set of real numbers under the substraction and scalar multiplication a vector space? Explain.

Solution:

NO. Clearly, for $x, y \in \mathbb{R}$, $x \oplus y = x - y \neq y - x = y \oplus x$.

Theorem 4.1.1

Let $\mathbb{V}$ be a vector space, X is a vector in $\mathbb{V}$, and c is a scalar, then:

1. The zero vector is unique in $\mathbb{V}$.
2. $0\,X = O$.
3. $c\,O = O$.
4. $(-1)X = -X$.
5. If $c\,X = O$, then $c = 0$ or $X = O$.

Proof:

1. Assume that O_1 and O_2 are two zero vectors in $\mathbb{V}$. Then
$$O_1 = O_1 + O_2 = O_2 \quad \Rightarrow \quad O_1 = O_2.$$
2. $0X + 0X = (0+0)X = 0X$. The negative inverse of $0X$, namely $-0X$, is in $\mathbb{V}$. Hence
$$0X + 0X + (-0X) = 0X + (-0X) \;\Rightarrow\; 0X = O.$$
3. $O + O = O$. Then $c(O + O) = cO$ implies that $cO + cO = cO$. Adding the negative of cO, namely $-cO$, to both sides, we get: $cO + cO - cO = cO - cO$ and hence $cO = O$.
4. To show that $(-1)X = -x$, we simply show that $X + (-1)X = O$. Clearly, $X + (-1)X = (1 + (-1))X = 0X = O$ by part (2).
5. If $c = 0$, then we are done. Otherwise, assume $c \neq 0$. Then $\frac{1}{c}\,c\,X = \frac{1}{c}\,O = O$ (by Part 3). Hence $1\,X = X = O$.

Exercise 4.1.1

1. Let $\mathbb{V} = \{(x, y, 0) \,|\, x, y \in \mathbb{R}\}$, be associated with the operations:

$$(x_1, y_1, 0) \oplus (x_2, y_2, 0) = (x_1 + x_2, y_1 + y_2, 0), \quad \text{and } c \odot (x, y, 0) = (cx, cy, 0).$$

Is $(\mathbb{V}, \oplus, \odot)$ form a vector space? Explain. (This is a Vector Space!!).

2. Let $V = \{(x, y) \mid x, y \in \mathbb{R}\}$. Define addition and scalar multiplication on V as follows: for each $(x, y), (x', y') \in V$ and $a \in \mathbb{R}$,

$$(x, y) \oplus (x', y') = (x + x', y + y') \quad \text{and} \quad a \odot (x, y) = (ay, ax).$$

Determine whether V with the given operations is a vector space. Justify your answer.

3. Consider $\mathbb{R}^2$ with the operations $\oplus$ and $\odot$ where $(x, y) \oplus (x', y') = (2x - x', 2y - y')$ and $c \odot (x, y) = c(x, y)$. Does the property $(c + d) \odot X = c \odot X \oplus d \odot X$ hold for all $c, d \in \mathbb{R}$ and all $X \in \mathbb{R}^2$? Explain.

4. Consider the set $V = \{(x, y, z) : x, y, z \in \mathbb{R}\}$ with the following operations

$$(x_1, y_1, z_1) \oplus (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2) \quad \text{and} \quad c \odot (x, y, z) = (z, y, x).$$

Is V a vector space? Explain.

5. Let $V = \{(x, y, z) : x, y, z \in \mathbb{R}\}$ and define

$$(x, y, z) \oplus (a, b, c) = (x + a, y + b, z + c) \quad \text{and} \quad k \odot (x, y, z) = (kx, ky, 0).$$

Show that V is not a vector space.

6. Consider the set $V = \{(x, y) \,|\, x, y \in \mathbb{R}\}$ with the following operations

$$(x, y) \oplus (x', y') = (x - x', y - y') \quad \text{and} \quad k \odot (x, y) = (kx, ky).$$

Determine whether $(V, \oplus, \odot)$ is a vector space (justify your answer).

7. Let V be a real vector space. Show that $0X = O$ for any $X \in V$.

8. Let V be a vector space with a zero vector O. Show that the zero vector O of V is unique.

9. Prove that the negative of a vector X in a vector space V is unique.

10. Determine whether $V = \mathbb{R}$ is a vector space with respect to the following operations: $X \oplus Y = 2X - Y$ and $c \odot X = cX$, for all $X, Y \in V$ and for all $c \in \mathbb{R}$.

11. Determine whether $V = \{(x,y,z) : x,y,z \in \mathbb{R}\}$ is a vector space with respect to the following operations: $(x,y,z) \oplus (x',y',z') = (xx',yy',zz')$ and $c \odot (x,y,z) = (cx,cy,cz)$, for all $(x,y,z),(x',y',z') \in V$ and for all $c \in \mathbb{R}$.
12. Answer each of the following as True or False:

    (a) (T) $V = \{(x,y) \in \mathbb{R}^2 : y < 0\}$ is closed under the operation $c \odot (x,y) = (cx,y)$.

    (b) (F) $V = \{(x,y) \in \mathbb{R}^2 : y < 0\}$ is closed under the operation $c \odot (x,y) = (cy,x)$.

4.2 Subspaces

Definition 4.2.1

Let $(\mathbb{V}, \oplus, \odot)$ be a vector space and let $\mathbb{W} \subseteq \mathbb{V}$ be non-empty. If $(\mathbb{W}, \oplus, \odot)$ is a vector space, then $\mathbb{W}$ is a **subspace** of $\mathbb{V}$.

Remark 4.2.1

If $\mathbb{V}$ is a vector space, then $\mathbb{V}$ and $\{0\}$ are subspaces of $\mathbb{V}$. They are called **trivial subspaces** of $\mathbb{V}$.

Theorem 4.2.1

Let $(\mathbb{V}, \oplus, \odot)$ be a vector space and let $\mathbb{W}$ be a subset of $\mathbb{V}$. Then, $\mathbb{W}$ is a subspace of $\mathbb{V}$ if and only if the following conditions hold:

1. $\mathbb{W} \neq \phi$,
2. for all $x, y \in \mathbb{W}$, $x \oplus y \in \mathbb{W}$,
3. for all $x \in \mathbb{W}$ and $c \in \mathbb{R}$, $c \odot x \in \mathbb{W}$.

Example 4.2.1

Is $\mathbb{W} = \{(x, y, 0, z^2) : x, y \in \mathbb{R} \text{ and } z \in \mathbb{Z}\}$ a subspace of $\mathbb{R}^4$? Explain.

Solution:

1. $(0,0,0,0) \in \mathbb{W}$ and hence $\mathbb{W}$ is non-empty.
2. let $(x_1, y_1, 0, z_1^2), (x_2, y_2, 0, z_2^2) \in \mathbb{W}$, then

$$(x_1, y_1, 0, z_1^2) + (x_2, y_2, 0, z_2^2) = (x_1 + x_2, y_1 + y_2, 0, z_1^2 + z_2^2) \notin \mathbb{W} \text{ since } z_1^2 + z_2^2 \neq (z_1 + z_2)^2.$$

For example, $(0,0,0,4), (0,0,0,9) \in \mathbb{W}$ while the sum of them $(0,0,0,15) \notin \mathbb{W}$. Therefore, $\mathbb{W}$ is not a subspace of $\mathbb{R}^4$.

Example 4.2.2

Is $\mathbb{W} = \{(x, y, z) : x + y + z = 1, \text{ where } x, y, z \in \mathbb{R}\}$ a subspace of $\mathbb{R}^3$? Explain.

Solution:

Clearly, $(0, 0, 0) \notin \mathbb{W}$ and hence $\mathbb{W}$ is not a vector space and it is not a subspace of $\mathbb{R}^3$.

Example 4.2.3

Let $\mathbb{W} = \{(a, b, c, d) : d = 2a - b \text{ and } c = a\}$. Is $(\mathbb{W}, +, \cdot)$ a subspace of $\mathbb{R}^4$? Explain.

Solution:

1. $(0, 0, 0, 0) \in \mathbb{W}$, then $\mathbb{W} \neq \phi$,
2. let $X = (a_1, b_1, a_1, 2a_1 - b_1)$ and $Y = (a_2, b_2, a_2, 2a_2 - b_2)$. Then,

$$\begin{aligned} X + Y &= (a_1, b_1, a_1, 2a_1 - b_1) + (a_2, b_2, a_2, 2a_2 - b_2) \\ &= (a_1 + a_2, b_1 + b_2, a_1 + a_2, 2(a_1 + a_2) - (b_1 + b_2)) \in \mathbb{W}. \end{aligned}$$

3. for $X = (a, b, a, 2a-b) \in \mathbb{W}$ and $k \in \mathbb{R}$, we have $k(a, b, a, 2a-b) = (ka, kb, ka, 2(ka), -kb) \in \mathbb{W}$.

Therefore, $\mathbb{W}$ is a subspace of $\mathbb{R}^4$.

Theorem 4.2.2

If $\mathbb{W}_1, \mathbb{W}_2, \cdots, \mathbb{W}_n$ are subspaces of a vector space $\mathbb{V}$, the the intersection of these subspaces is also a subspace of $\mathbb{V}$.

Proof:

Let $\mathbb{W}$ be the intersection of these subspaces. Then $\mathbb{W}$ is not empty since $\mathbb{W}_i$ contains the zero vector for all $1 \le i \le n$. Moreover, if $X, Y \in \mathbb{W}$, then $X, Y \in \mathbb{W}_i$ for all i and hence $X + Y \in \mathbb{W}_i$ which implies that $X + Y \in \mathbb{W}$. Finaly, if c is a scalar and $X \in \mathbb{W}$, then $X \in \mathbb{W}_i$ for all i and hence $c\,X \in \mathbb{W}_i$ which implies that $c\,X \in \mathbb{W}$. Therefore, $\mathbb{W}$ is a subspace of $\mathbb{V}$.

Definition 4.2.2

Let $AX = O$ be a homogenous system for $A \in M_{m\times n}$ and $X \in \mathbb{R}^n$. We define the **null space** (or the solution space of $AX = O$) of A by

$$\mathbb{W} = \{X : AX = O\} \subseteq \mathbb{R}^n.$$

Theorem 4.2.3

The solution space of the homogeneous system $AX = O$ (null space of A), where A is $m \times n$ matrix and $X \in \mathbb{R}^n$ is a subspace of $\mathbb{R}^n$.

Proof:

Let $\mathbb{W} = \{X : AX = O\} \subseteq \mathbb{R}^n$ be the null space of A. Then

1. Clearly, $\mathbb{W} \neq \phi$ since $AX = O$ always has a solution (either trivial or non-trivial),
2. If $X, Y \in \mathbb{W}$, then $AX = AY = O$. But, $A(X+Y) = AX + AY = O + O = O$. Thus, $X + Y \in \mathbb{W}$,
3. For any $c \in \mathbb{R}$ and $X \in \mathbb{W}$, we have $A(cX) = cAX = cO = O$, thus $cX \in \mathbb{W}$.

Therefore, $\mathbb{W}$ is a subspace of $\mathbb{R}^n$.

If $AX = B$ with $B \neq O$ and $A \in M_{m\times n}$ and $X \in \mathbb{R}^n$. Then $\mathbb{W} = \{X : AX = B\} \subseteq \mathbb{R}^n$ is not necessarily a subspace of $\mathbb{R}^n$. Simply, the system $AX = B$ might have no solutions and then $\mathbb{W}$ is empty.

Definition 4.2.3

Let $X_1, X_2, \cdots, X_n$ be vectors in a vector space $\mathbb{V}$, a vector $X \in \mathbb{V}$ is called a **linear combination** of the vectors $X_1, X_2, \cdots, X_n$ if and only if $X = c_1X_1 + c_2X_2 + \cdots + c_nX_n$ for some real numbers $c_1, c_2, \cdots, c_n$.

Example 4.2.4

Determine whether the vector $X = (2, 1, 5)$ is a linear combination of the set of vectors $\{X_1, X_2, X_3\}$ where $X_1 = (1, 2, 1)$, $X_2 = (1, 0, 2)$, and $X_3 = (1, 1, 0)$.

Solution:

X is a linear combination of X_1, X_2, X_3 if we find numbers c_1, c_2, c_3 so that $X = c_1X_1 + c_2X_2 + c_3X_3$. Consider $c_1(1, 2, 1) + c_2(1, 0, 2) + c_3(1, 1, 0) = (2, 1, 5)$, this is a system in three unknowns:

$$\begin{aligned} c_1 + c_2 + c_3 &= 2 \\ 2c_1 + 0 + c_3 &= 1 \\ c_1 + 2c_2 + 0 &= 5 \end{aligned}$$

So, we solve the system:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 2 & 0 & 1 & 1 \\ 1 & 2 & 0 & 5 \end{array}\right] \xrightarrow[r_3 - r_1 \to r_3]{r_2 - 2r_1 \to r_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -2 & -1 & -3 \\ 0 & 1 & -1 & 3 \end{array}\right] \xrightarrow{r_2 \Leftrightarrow r_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & -1 & 3 \\ 0 & -2 & -1 & -3 \end{array}\right] \xrightarrow[r_3 + 2r_2 \to r_3]{r_1 - r_2 \to r_1}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & -1 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & -3 & 3 \end{array}\right] \xrightarrow{-\frac{1}{3}r_3 \to r_3} \left[\begin{array}{ccc|c} 1 & 0 & 2 & -1 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & -1 \end{array}\right] \xrightarrow[r_2 + r_3 \to r_2]{r_1 - 2r_3 \to r_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

Therefore, $\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ is a solution. Thus, $X = X_1 + 2X_2 - X_3$ (check!).

Note that we can solve the problem as follows: The matrix of coefficient above has determinant equals to 3 and hence the system has a unique solution. Therefore there are c_1, c_2, and c_3 satisfying the linear combination equation. This show that X is a linear combination of X_1, X_2, X_3 and we are done without solving the system.

Definition 4.2.4

Let $S = \{X_1, X_2, \cdots, X_k\}$ be a subset of a vector space $\mathbb{V}$. Then, the set of all vectors in $\mathbb{V}$ that are linear combination of the vectors in S is denoted by **span** S or **span** $\{X_1, X_2, \cdots, X_k\}$. Moreover, if $\mathbb{W} =$ **span** S then $\mathbb{W}$ is a subspace of $\mathbb{V}$ and we say that S **spans** $\mathbb{W}$ or that $\mathbb{W}$ is **spanned by** S.

Theorem 4.2.4

If $S = \{X_1, X_2, \cdots, X_k\}$ is a nonempty set of vectors in a vector space $\mathbb{V}$, then **span** S is a subspace of $\mathbb{V}$.

Proof:

Let $\mathbb{W} =$ **span** $S = \{Z : Z = c_1X_1 + c_2X_2 + \cdots + c_kX_k\} \subseteq \mathbb{V}$. Then,

1. $\mathbb{W} \neq \phi$ since $Z = X_1 + X_2 + \cdots + X_k \in \mathbb{W}$ for $c_i = 1$ for all $i = 1, 2, \cdots, k$,
2. let $Z_1 = c_1X_1 + c_2X_2 + \cdots + c_kX_k$, $Z_2 = d_1X_1 + d_2X_2 + \cdots + d_kX_k \in \mathbb{W}$, then

$$Z_1 + Z_2 = (c_1 + d_1)X_1 + (c_2 + d_2)X_2 + \cdots + (c_k + d_k)X_k \in \mathbb{W},$$

3. for $c \in \mathbb{R}$ and $Z = c_1X_1 + c_2X_2 + \cdots + c_kX_k \in \mathbb{W}$, we have

$$Z = c\,c_1X_1 + c\,c_2X_2 + \cdots + c\,c_kX_k \in \mathbb{W}.$$

Therefore, $\mathbb{W} =$ **span** S is a subspace of $\mathbb{V}$.

Theorem 4.2.5

If S is a nonempty set of vectors in a vector space $\mathbb{V}$, then **span** S is the smallest subspace of $\mathbb{V}$ that contains all of the vectors in S. That is, any other subspace that contains S contains **span** S.

Example 4.2.5

Let $S = \{(1, 1, 0, 1), (1, -1, 0, 1), (0, 1, 2, 1)\}$. Determine whether X and Y belong to **span** S, where $X = (2, 3, 2, 3)$, and $Y = (0, 1, 2, 3)$.

Solution:

For X, consider the system $X = (2, 3, 2, 3) = c_1(1, 1, 0, 1) + c_2(1, -1, 0, 1) + c_3(0, 1, 2, 1)$. This system has the unique solution: $c_1 = 2, x_2 = 0, c_3 = 1$. Thus, X belongs to **span** S.
For Y we consider the system $Y = (0, 1, 2, 3) = c_1(1, 1, 0, 1) + c_2(1, -1, 0, 1) + c_3(0, 1, 2, 1)$. This system is inconsistent and has no solutions. Thus, Y does not belong to **span** S.

Recall that any vector in $\mathbb{R}^n$ can be written as a linear combination of the standard unit vectors $\{E_1, E_2, \cdots, E_n\}$. Thus, $\{E_1, E_2, \cdots, E_n\}$ spans $\mathbb{R}^n$.

We now go back to show how can we solve Example 4.2.3 by three different method. One method was already shown in that example.

Example 4.2.6

Let $\mathbb{W} = \{(a, b, c, d) : d = 2a - b \text{ and } c = a\}$. Is $(\mathbb{W}, +, \cdot)$ a subspace of $\mathbb{R}^4$? Explain.

Solution: Definition

We simply show it by the meaning of the definition of subspaces. Look at the Example 4.2.3.

Solution: Null Space

$$\mathbb{W} = \{(a, b, c, d)\ : 2a - b - d = 0 \text{ and } a - c = 0\}.$$

That is $\mathbb{W}$ is the solution space of $AX = O$ where $A = \begin{bmatrix} 2 & -1 & 0 & -1 \\ 1 & 0 & -1 & 0 \end{bmatrix}$ and $X = (a, b, c, d)$. Therefore, $\mathbb{W}$ is a subspace of $\mathbb{R}^4$.

Solution: The Span

$$\mathbb{W} = \{(a, b, a, 2a - b)\ : a, b \in \mathbb{R}\} = \{a\,(1, 0, 1, 2) + b\,(0, 1, 0, -1)\ : a, b \in \mathbb{R}\}.$$

Therefore $\mathbb{W} = \textbf{span}\ \{(1, 0, 1, 2), (0, 1, 0, -1)\}$. That is $\mathbb{W}$ is a subspace of $\mathbb{R}^4$.

Example 4.2.7

Let $S = \{X_1, X_2\}$ where $X_1 = (1, 1, 0)$, and $X_2 = (1, 1, 1)$. Does S spans $\mathbb{R}^3$? Explain.

Solution:

Let $X = (a, b, c)$ be any vector in $\mathbb{R}^3$. Consider

$$c_1(1, 1, 0) + c_2(1, 1, 1) = (a, b, c).$$

Note that we can not use the determinant argument here since we have no square matrix. Thus, solving the system, we get

$$\left[\begin{array}{cc|c} 1 & 1 & a \\ 1 & 1 & b \\ 0 & 1 & c \end{array}\right] \xrightarrow{r_2 - r_1 \to r_2} \left[\begin{array}{cc|c} 1 & 1 & a \\ 0 & 0 & b-a \\ 0 & 1 & c \end{array}\right].$$

This system has no solution if $b - a \neq 0$. Therefore, X is not a linear combination of S and S does not span $\mathbb{R}^3$.

Example 4.2.8

Let $S = \{X_1, X_2\}$ where $X_1 = (1,1,0)$, and $X_2 = (1,1,1)$. Does S spans $\mathbb{R}^3$? Explain.

Solution:

Let $X = (a, b, c)$ be any vector in $\mathbb{R}^3$. Consider

$$c_1(1,1,0) + c_2(1,1,1) = (a, b, c).$$

Note that we can not use the determinant argument here since we have no square matrix. Thus, solving the system, we get

$$\left[\begin{array}{cc|c} 1 & 1 & a \\ 1 & 1 & b \\ 0 & 1 & c \end{array}\right] \xrightarrow{r_2 - r_1 \to r_2} \left[\begin{array}{cc|c} 1 & 1 & a \\ 0 & 0 & b-a \\ 0 & 1 & c \end{array}\right].$$

This system has no solution if $b - a \neq 0$. Therefore, X is not a linear combination of S and S does not span $\mathbb{R}^3$.

Exercise 4.2.1

1. Let $X, Y \in \mathbb{R}^n$ and $\mathbb{W} = \{Z : Z = aX + bY,\ \text{for } a, b \in \mathbb{R}\}$. Is $\mathbb{W}$ a subspace of $\mathbb{R}^n$? Explain.
2. Let $X_1 = (1, 0, 2)$, $X_2 = (2, 0, 1)$ and $X_3 = (1, 0, 3)$ be vectors in $\mathbb{R}^3$. Determine whether the vector $X = (1, 2, 3)$ can be written as a linear combination of X_1, X_2, and X_3.
3. Determine whether the subset $\mathbb{W} = \left\{\left(a, a + \sqrt{2}, 3a\right)\ :\ a \in \mathbb{R}\right\}$ of $\mathbb{R}^3$ is a subspace of $\mathbb{R}^3$.
4. Determine whether the vectors $X_1 = (3, 0, 0, 0)$, $X_2 = (0, -1, 2, 1)$, $X_3 = (6, 2, -6, 0)$, and $X_4 = (3, -2, 3, 3)$ spans the vector space $\mathbb{R}^4$.
5. Show that the solution set of a homogeneous linear system $AX = O$ in n unknowns is a subspace of $\mathbb{R}^n$.
6. Let $\mathbb{W} = \{(a, 2a, b, a - b)\ :\ a, b \in \mathbb{R}\}$ be a subset of $\mathbb{R}^4$. Show that $\mathbb{W}$ is a subspace of $\mathbb{R}^4$.
7. Determine whether $\mathbb{W}_1$ and $\mathbb{W}_2$ are subspaces of $\mathbb{R}^4$.
   (a) $\mathbb{W}_1 = \{(a, b, c, d)\ :\ a^2 + b^2 + c^2 + d^2 > 0\}$.
   (b) $\mathbb{W}_2 = \{(a, b, c, d)\ :\ a + 3b - 2c + 4d = 0 \text{ and } a - 5b + 4c + 7d = 0\}$.
8. Determine whether $\mathbb{W}_1$ and $\mathbb{W}_2$ are subspaces of $\mathbb{R}^3$.
   (a) $\mathbb{W}_1 = \{(a, b, c)\ :\ a - c = b\}$.
   (b) $\mathbb{W}_2 = \{(a, b, c)\ :\ ab \geq 0\}$.

4.3 Linear Independence

Definition 4.3.1

The set of vectors $S = \{X_1, X_2, \cdots, X_n\}$ in a vector space $\mathbb{V}$ is said to be **linearly dependent** if there exist constants $c_1, c_2, \cdots, c_n$ not all zeros, such that

$$c_1X_1 + c_2X_2 + \cdots + c_nX_n = \mathbf{0}.$$

Otherwise, S is said to be **linearly independent**. That is, $X_1, X_2, \cdots, X_n$ are linearly independent if whenever, $c_1X_1 + c_2X_2 + \cdots + c_nX_n = \mathbf{0}$, we must have $c_1 = c_2 = \cdots = c_n = 0$.

Note that the standard unit vectors are linearly independent in $\mathbb{R}^n$ since the homogeneous system

$$c_1(1, 0, \cdots, 0) + c_2(0, 1, \cdots, 0) + \cdots + c_n(0, 0, \cdots, 1) = (0, 0, \cdots, 0)$$

clearly has only the trivial solution $c_1 = c_2 = \cdots = c_n = 0$.

Example 4.3.1

Determine whether $X_1 = (1, 0, 1, 2)$, $X_2 = (0, 1, 1, 2)$, and $X_3 = (1, 1, 1, 3)$ in $\mathbb{R}^4$ are linearly independent or linearly dependent? Explain.

Solution:

We solve the homogenous system: $c_1X_1 + c_2X_2 + c_3X_3 = \mathbf{0}$. That is,

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 2 & 2 & 3 & 0 \end{array}\right] \to \cdots \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore, $c_1 = c_2 = c_3 = 0$ and thus X_1, X_2, X_3 are linearly independent.

Example 4.3.2

Determine whether the vectors $X_1 = (1, 2, -1)$, $X_2 = (1, -2, 1)$, $X_3 = (-3, 2, -1)$, and $X_4 = (2, 0, 0)$ in $\mathbb{R}^3$ is linearly independent or linearly dependent? Explain.

Solution:

Consider the homogenous system: $c_1X_1 + c_2X_2 + c_3X_3 + c_4X_4 = \mathbf{0}$. This system has a non-trivial solutions because the number of unknowns (4) is greater than the number of equations (3). Therefore, X_1, X_2, X_3, X_4 are linearly dependent.

In addition, we can show that this set is linearly dependent by the mean of r.r.e.f. as follows:

$$\left[\begin{array}{cccc|c} 1 & 1 & -3 & 2 & 0 \\ 2 & -2 & 2 & 0 & 0 \\ -1 & 1 & -1 & 0 & 0 \end{array}\right] \to \cdots \to \left[\begin{array}{cccc|c} 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

From the reduced system above, we see that (from the third column) $X_3 = -X_1 - 2X_2$ and that (from the fourth column) $X_4 = X_1 + X_2$.

Theorem 4.3.1

A set S with two or more vectors is

1. Linearly dependent iff at least one of the vectors in S is a linear combination of the other vectors in S.
2. Linearly independent iff no vectors in S is a linear combination of the other vectors in S.

Remark 4.3.1

Let $S = \{X_1, X_2, \cdots, X_n\}$ be a set of vectors in $\mathbb{R}^n$ and let A be an $n \times n$ matrix whose columns are the n-vectors of S. Then,

1. if A is singular, then S is linearly dependent,
2. if A is non-singular, then S is linearly independent.

Theorem 4.3.2

1. A set that contains O "the zero vector" is linearly dependent.
2. A set with exactly one vector is linearly independent iff that vector is not O.
3. A set with exactly two nonzero vectors is linearly independent iff neither vector is a scalar multiple of the other.

Example 4.3.3

For what values of α are the vectors $(-1,0,-1),(2,1,2),(1,1,\alpha)$ in $\mathbb{R}^3$ linearly dependent? Explain.

Solution:

We want the vectors to be linearly dependent, so consider the system $c_1(-1,0,-1)+c_2(2,1,2)+c_3(1,1,\alpha)=(0,0,0)$. This system has non-trivial solutions only if $\left|A\right|=0$, where A is the matrix whose columns are $[-1,0,-1]^T,[2,1,2]^T$, and $[1,1,\alpha]^T$. That is,

$$\left|A\right|=\begin{vmatrix} -1 & 2 & 1 \\ 0 & 1 & 1 \\ -1 & 2 & \alpha \end{vmatrix}=0 \iff -(\alpha-2)-(2-1)=0 \iff 2-\alpha-1=0 \iff \alpha=1.$$

Therefore, if $\alpha=1$ the vectors are linearly dependent. Otherwise if $\alpha\in\mathbb{R}\backslash\{1\}$, the vectors are linearly independent.

Theorem 4.3.3

Let $S=\{X_1,X_2,\cdots,X_m\}$ be a set of vectors in $\mathbb{R}^n$. If $m>n$, then S is linearly dependent.

Example 4.3.4

Suppose that $S=\{X_1,X_2,X_3\}$ is a linearly independent set of vectors in a vector space $\mathbb{V}$. Show that $T=\{Y_1,Y_2,Y_3\}$ is also linearly independent set, where $Y_1=X_1+X_2+X_3$, $Y_2=X_2+X_3$, and $Y_3=X_3$

Solution:

Consider the system $c_1Y_1 + c_2Y_2 + c_3Y_3 = \mathbf{0}$. Therefore,

$$\begin{array}{rcrcrcl} c_1Y_1 & + & c_2Y_2 & + & c_3Y_3 & = & \mathbf{0} \\ c_1(X_1+X_2+X_3) & + & c_2(X_2+X_3) & + & c_3(X_3) & = & \mathbf{0} \\ c_1X_1+c_1X_2+c_1X_3 & + & c_2X_2+c_2X_3 & + & c_3X_3 & = & \mathbf{0} \\ (c_1)X_1 & + & (c_1+c_2)X_2 & + & (c_1+c_2+c_3)X_3 & = & \mathbf{0} \end{array}$$

But X_1, X_2, X_3 are linearly independent, thus $c_1 = c_1 + c_2 = c_1 + c_2 + c_3 = 0$. Therefore, $c_1 = c_2 = c_3 = 0$ and hence T is linealry independent.

Example 4.3.5

Suppose that $S = \{X_1, X_2, X_3\}$ is a linearly dependent set of vectors in a vector space $\mathbb{V}$. Show that $T = \{Y_1, Y_2, Y_3\}$ is also linearly dependent set, where $Y_1 = X_1$, $Y_2 = X_1 + X_2$, and $Y_3 = X_1 + X_2 + X_3$

Solution:

Consider the system $c_1Y_1 + c_2Y_2 + c_3Y_3 = \mathbf{0}$. Therefore,

$$\begin{array}{rcrcrcl} c_1Y_1 & + & c_2Y_2 & + & c_3Y_3 & = & \mathbf{0} \\ c_1(X_1) & + & c_2(X_1+X_2) & + & c_3(X_1+X_2+X_3) & = & \mathbf{0} \\ (c_1+c_2+c_3)X_1 & + & (c_2+c_3)X_2 & + & (c_3)X_3 & = & \mathbf{0} \end{array}$$

But X_1, X_2, X_3 are linearly dependent, thus at least one of $c_1+c_2+c_3$, c_2+c_3, and c_3 is non-zero. Therefore, one of c_1, c_2, c_3 is non-zero and hence T is linealry dependent.

Remark 4.3.2

The set of linearly independent vectors should be non-zeros distinct vectors.

Remark 4.3.3

Let S_1, S_2 be two subsets of a vector space $\mathbb{V}$ with $S_1 \subseteq S_2$. Then,

1. if S_1 is linearly dependent, then S_2 is linearly dependent,
2. if S_2 is linearly independent, then S_1 is linearly independent.

Example 4.3.6

Show that if $S = \{X_1, X_2, \cdots, X_n\}$ is a linearly independent set of vectors, then so is any subset of S.

Solution:

Let $T = \{X_1, \cdots, X_k\}$ with $k \leq n$. As S is linearly independent set, $c_1X_1 + \cdots c_nX_n = O$ implies that $c_1 = \cdots = c_n = 0$. To show that T is linearly independent, we show that $c_1 = \cdots = c_k = 0$. Consider the system $c_1X_1 + \cdots + c_kX_k = O$. Then

$$c_1X_1 + c_2X_2 + \cdots + c_kX_k + (0)X_{k+1} + \cdots + (0)X_n = O.$$

This system has only the trivial solution and hence $c_1 = \cdots = c_k = 0$. Therefore, T is linealy independent.

Note that if $S = \{X_1, X_2, X_3\}$ is a linear independent set, then as we have seen in the previous example the sets $\{X_1, X_2\}$, $\{X_1, X_3\}$, $\{X_2, X_3\}$, $\{X_1\}$, $\{X_2\}$, and $\{X_3\}$ are also linearly independent.

Example 4.3.7

Show that if $S = \{X_1, X_2, X_3\}$ is a linearly dependent set of vectors in a vector space $\mathbb{V}$, and X_4 is any vector in $\mathbb{V}$ that is not in S, then $\{X_1, X_2, X_3, X_4\}$ is also linearly dependent.

Solution:

Since S is linealy dependent set, the homogeneous system $c_1X_1 + c_2X_2 + c_3X_3 = O$ has a non trivial solution (c_1, c_2, c_3). Then the homogeneous system $c_1X_1 + c_2X_2 + c_3X_3 + c_4X_4 = O$ also has a nontrivial solution $(c_1, c_2, c_3, 0)$.

Exercise 4.3.1

1. Show that if $\{X_1, X_2\}$ is a linearly dependent set, then one of the vector is a scalar multiple of the other.
2. Show that any subset of a vector space $\mathbb{V}$ contains the zero vector is a linearly dependent set.
3. Show that if $\{X_1, X_2, \cdots, X_n\}$ is a linearly dependent set, then we can express one of the vectors in terms of the others.
4. Let $X, Y, Z \in \mathbb{R}^n$ be three nonzero vectors where the dot product of any (distinct) two vectors is 0. Show that the set $\{X, Y, Z\}$ is linearly independent.

4.4 Basis and Dimension

Definition 4.4.1

A set $S = \{X_1, X_2, \cdots, X_n\}$ of distinct nonzero vectors in a vector space $\mathbb{V}$ is called a **basis** iff

1. S spans $\mathbb{V}$ ($\mathbb{V} = \text{span } S$),
2. S is linearly independent set.

The **dimension** of $\mathbb{V}$ is the number of vectors in its basis and is denoted by $\mathbf{dim}(\mathbb{V})$.

Example 4.4.1

Show that the set $S = \{X_1 = (1,0,1), X_2 = (0,1,-1), X_3 = (0,2,2)\}$ is a basis for $\mathbb{R}^3$.

Solution:

To show that S is a basis for $\mathbb{R}^3$, we show that S is a linearly independent set that spans $\mathbb{R}^3$.

1. S is linealy independent? Consider the homogenous system

$$c_1(1,0,1) + c_2(0,1,-1) + c_3(0,2,2) = (0,0,0).$$

This system has a trivial solution if $\left|A\right| \neq 0$, where A is the matrix of coefficients. That is,

$$|A| = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 1 & -1 & 2 \end{vmatrix} = (1)\begin{vmatrix} 1 & 2 \\ -1 & 2 \end{vmatrix} = 2 - (-2) = 4 \neq 0.$$

Thus, the system has only the trivial solution and hence S is linearly independent.

2. S spans $\mathbb{R}^3$? For any $X = (a,b,c) \in \mathbb{R}^3$, consider the nonhomogenous system:

$$c_1(1,0,1) + c_2(0,1,-1) + c_3(0,2,2) = (a,b,c).$$

Since the $\left|A\right| \neq 0$ where A is the matrix of coefficients, the system has a unique solution and thus S spans $\mathbb{R}^3$.

Therefore, S is a basis for $\mathbb{R}^3$.

Remark 4.4.1

The set of standard unit vectors $\{\mathbf{E}_1, \mathbf{E}_2, \cdots, \mathbf{E}_n\} \in \mathbb{R}^n$ forms the standard basis for $\mathbb{R}^n$ and hence $\mathbf{dim}(\mathbb{R}^n) = n$.

Theorem 4.4.1

Let $S = \{X_1, X_2, \cdots, X_n\}$ be a basis for a vector space $\mathbb{V}$. Then, every vector in $\mathbb{V}$ can be written in exactly one way as a linear combination of the vectors in S.

Proof:

Let $X \in \mathbb{V}$. Since S is a basis of $\mathbb{V}$, then S spans $\mathbb{V}$. That is, we can write

$$\begin{aligned} X &= c_1X_1 + c_2X_2 + \cdots + c_nX_n, \text{ and} && (4.4.1)\\ X &= d_1X_1 + d_2X_2 + \cdots + d_nX_n. && (4.4.2) \end{aligned}$$

By substracting Equation 4.4.2 out of Equation 4.4.1, we get

$$\mathbf{0} = (c_1 - d_1)X_1 + (c_2 - d_2)X_2 + \cdots + (c_n - d_n)X_n.$$

But S is linearly independent set (it is a basis). Thus, $c_1 - d_1 = 0, \ \cdots, \ c_n - d_n = 0$. Therefore, $c_1 = d_1, \ c_2 = d_2, \ \cdots, \ c_n = d_n$, and hence X can be written in one and only one way as a linear combination of vectors in S.

Theorem 4.4.2

Let $\mathbb{V}$ be a finite-dimensional vector space, and let $\{X_1, X_2, \cdots, X_n\}$ be any basis:

1. If a set has more than n vectors, then it is linearly dependent.
2. If a set has fewer than n vectors, then it does not span $\mathbb{V}$.

Theorem 4.4.3

All bases for a finite-dimensional vector space have the same number of vectors.

Theorem 4.4.4: The Plus/Minus Theorem

Let S be a nonempty set of vectors in a vector space $\mathbb{V}$. Then

1. If S is linearly independent and X is a vector in $\mathbb{V}$ not in **span** S, then the set $S \cup \{X\}$ is linearly independent.
2. If X is a vector in S that is a linear combination of the other vectors in S, then $S - \{X\}$ span the same space. That is, **span** $S =$ **span** $(S - \{X\})$.

Example 4.4.2

Find a basis for and the dimension of the subspace of all vectors of the form $(a, b, -a-b, a-b)$, for $a, b \in \mathbb{R}$.

Solution:

Let $\mathbb{W} = \{(a, b, -a-b, a-b) \,|\, a, b \in \mathbb{R}\} \subseteq \mathbb{R}^4$. Let X be any vector of $\mathbb{W}$, then

$$X = (a, b, -a-b, a-b) = a(1, 0, -1, 1) + b(0, 1, -1, -1) \in \mathbb{W}.$$

Therefore, $S = \{(1, 0, -1, 1), (0, 1, -1, -1)\}$ spans $\mathbb{W}$.

Clearly,

$$c_1(1, 0, -1, 1) + c_2(0, 1, -1, -1) = (0, 0, 0, 0)$$

holds only if $c_1 = c_2 = 0$ which shows that S is linearly independent. That is, S is a basis for $\mathbb{W}$ and $\mathbf{dim}(\mathbb{W}) = 2$.

Example 4.4.3

Find a basis for and the dimension of the solution space of the homogeneous system

$$\begin{array}{ccccccccc} x_1 & + & x_2 & & & + & 2x_4 & = & 0 \\ & & x_2 & - & x_3 & + & x_4 & = & 0 \\ x_1 & + & x_2 & & & + & 2x_4 & = & 0 \end{array}$$

Solution:

We first solve the system using Gauss-Jordan method:

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 1 & 1 & 0 & 2 & 0 \end{array}\right] \approx \overset{\text{r.r.e.f.}}{\cdots\cdots} \approx \left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Thus the solutions are: $x_1 = -x_3 - x_4$; $x_2 = x_3 - x_4$ and $x_3 = t$ and $x_4 = r$ for $t, r \in \mathbb{R}$. That is the solution space of the homogeneous system is $\mathbb{W} = \{(-t-r, t-r, t, r) : t, r \in \mathbb{R}\}$.
Therefore, any vector X in $\mathbb{W}$ is of the form: $X = t(-1, 1, 1, 0) + r(-1, -1, 0, 1)$ which means that $S = \{(-1, 1, 1, 0), (-1, -1, 0, 1)\}$ spans $\mathbb{W}$.
As S is a linearly independent set (none of the vectors is a scalar multiple of the other), S forms a basis for $\mathbb{W}$, and hence the solution space has dimension 2.

Theorem 4.4.5

Let $\mathbb{V}$ be an n-dimensional vector space, and let $S = \{X_1, X_2, \cdots, X_n\}$ be a set in $\mathbb{V}$. Then, S is a basis for $\mathbb{V}$ iff S spans $\mathbb{V}$ or S is linearly independent.

The set $S = \{X_1 = (1, 5), X_2 = (1, 4)\}$ is linear independent in the 2-dimensional vector space $\mathbb{R}^2$. Hence, S forms a basis for $\mathbb{R}^2$.
Moreover, considering $S = \{X_1 = (1, 0, 5), X_2 = (1, 0, 4), X_3 = (1, 1, 1)\}$, we see that X_1 and X_2 form a linear independent set in the xz-plane. The vector X_3 is outside of the xz-plane, so the set S is linearly independent set in $\mathbb{R}^3$. Hence, S forms a basis for $\mathbb{R}^3$.

Example 4.4.4

Find all values of a for which $S = \{(a^2, 0, 1), (0, a, 2), (1, 0, 1)\}$ is a basis for $\mathbb{R}^3$.

Solution:

Since $\mathbf{dim}(\mathbb{R}^3) = 3 =$ size of S, it is enough to show that S is linearly independent (or it spans $\mathbb{R}^3$) to show that it is a basis for $\mathbb{R}^3$. Consider $c_1(a^2, 0, 1) + c_2(0, a, 2) + c_3(1, 0, 1) = (0, 0, 0)$.

Clearly, S is linearly independent if $\left|A\right| \neq 0$, where A is the coefficient matrix. That is,

$$\begin{vmatrix} a^2 & 0 & 1 \\ 0 & a & 0 \\ 1 & 2 & 1 \end{vmatrix} = a(a^2-1) \neq 0 \quad \Longrightarrow \quad a \neq 0 \text{ and } a \neq \pm 1.$$

Therefore, S is a basis for $\mathbb{R}^3$ if $a \in \mathbb{R}\backslash\{-1,0,1\}$.

Theorem 4.4.6: Reduction and Extension Theorem

Let S be a finite set of vectors in a finite-dimensional vector space $\mathbb{V}$.

1. If S spans $\mathbb{V}$ but is not a basis, then S can be reduced to a basis for $\mathbb{V}$ by removing appropriate vectors from S.
2. If S is a linearly independent set that is not a basis for $\mathbb{V}$, then S can be extended to a basis for $\mathbb{V}$ by adding appropriate vectors to S.

Remark 4.4.2: How to construct a basis?

Let $\mathbb{V}$ be a vector space and $S = \{X_1, X_2, \cdots, X_n\}$ is a subset of $\mathbb{V}$. The procedure to find a subset of S that is a basis for $\mathbb{W} = \textbf{span } S$ is:

1. form the linear combination $c_1X_1 + c_2X_2 + \cdots + c_nX_n = \mathbf{0}$,
2. form the augmented matrix of the homogenous system in step (1),
3. find the r.r.e.f. of the augmented matrix,
4. Vectors in S corresponding to leading columns form a basis for $\mathbb{W} = \textbf{span } S$.

Example 4.4.5

Let $S = \{X_1 = (1,0,1), X_2 = (1,1,1), X_3 = (0,-1,0), X_4 = (2,1,2)\}$ be a set of vectors in $\mathbb{R}^3$. Find a subset of S that is a basis for $\mathbb{W} = \textbf{span } S$, and find the dimension of $\mathbb{W}$.

Solution:

We form the homogenous system: $c_1X_1 + c_2X_2 + c_3X_3 + c_4X_4 = O$ to find a linearly independent subset of S:

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 1 & 1 & 0 & 2 & 0 \end{array}\right] \approx \overset{\text{r.r.e.f.}}{\cdots\cdots} \approx \left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The leading entries are pointing (appear) on the first two columns, namely columns 1 and 2. Therefore, $\{X_1, X_2\}$ is linearly independent and it spans $\mathbb{W}$. Thus, $\{X_1, X_2\}$ is a basis for $\mathbb{W} = \textbf{span } S$ and $\textbf{dim}(\mathbb{W}) = 2$.

Example 4.4.6

Find a basis for $\mathbb{R}^4$ that contains the vectors $X_1 = (1, 0, 1, 0)$ and $X_2 = (-1, 1, -1, 0)$.

Solution:

Consider the set $S = \{X_1, X_2, \mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3, \mathbf{E}_4\}$. The set S spans $\mathbb{R}^4$ but it contains some linearly dependent vectors. In order to delete those, we follow the following procedure:

$$\left[\begin{array}{cccccc|c} 1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \approx \overset{\text{r.r.e.f.}}{\cdots\cdots} \approx \left[\begin{array}{cccccc|c} 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

The leading entries pointing on the columns 1, 2, 3, and 6. Therefore, the set $\{X_1, X_2, \mathbf{E}_1, \mathbf{E}_4\}$ is a basis for $\mathbb{R}^4$ containing X_1 and X_2.

Theorem 4.4.7

If $\mathbb{W}$ is a subspace of a finite-dimensional vector space $\mathbb{V}$, then

1. $\mathbb{W}$ is finite-dimensional.
2. $\textbf{dim}(\mathbb{W}) \le \textbf{dim}(\mathbb{V})$.
3. $\mathbb{W} = \mathbb{V}$ iff $\textbf{dim}(\mathbb{W}) = \textbf{dim}(\mathbb{V})$.

Exercise 4.4.1

1. Show that the set $S = \{X_1, X_2, X_3, X_4\}$ is a basis for $\mathbb{R}^4$, where
$$X_1 = (1,0,1,0), X_2 = (0,1,-1,2), X_3 = (0,2,2,1), \text{ and } X_4 = (1,0,0,1).$$
2. Let $S = \{X_1, X_2, X_3, X_4, X_5\}$ be a set of $\mathbb{R}^4$ where $X_1 = (1,2,-2,1)$, $X_2 = (-3,0,-4,3)$, $X_3 = (2,1,1,-1)$, $X_4 = (-3,3,-9,6)$, and $X_5 = (9,3,7,-6)$. Find a subset of S that is a basis for $\mathbb{W} = \textbf{span } S$. Find $\textbf{dim}(\mathbb{W})$. Final answer: $\{X_1, X_2\}$ is a basis for $\mathbb{W}$ and the dimension is 2.
3. Find the dimension of the subspace of all vectors of the form (a,b,c,d) where $c = a - b$ and $d = a + b$ (for $a, b \in \mathbb{R}$). Final answer: the dimension is 2.
4. Find the dimension of the subspace of all vectors of the form $(a+c, a+b+2c, a+c, a-b)$ where $a, b, c \in \mathbb{R}$. Final answer: the dimension is 2.
5. Let $S = \{X_1, X_2, X_3\}$ be a basis for a vector space $\mathbb{V}$. Show that $T = \{Y_1, Y_2, Y_3\}$ is also a basis for $\mathbb{V}$, where $Y_1 = X_1 + X_2 + X_3$, $Y_2 = X_2 + X_3$, and $Y_3 = X_3$.
6. Find a standard basis vector for $\mathbb{R}^3$ that can be added to the set $\{X_1 = (1,1,1), X_2 = (2,-1,3)\}$ to produce a basis a basis for $\mathbb{R}^3$. Final answer: any vector of the standard basis will work.
7. The set $S = \{X_1 = (1,2,3), X_2 = (0,1,1)\}$ is linearly independent in $\mathbb{R}^3$. Extend (enlarge) S to a basis for $\mathbb{R}^3$. Final answer: $S = \{X_1 = (1,2,3), X_2 = (0,1,1), X_3 = (1,0,0)\}$
8. Let $S = \{X_1 = (1,0,2), X_2 = (-1,0,-1)\}$ be a set of vectors in $\mathbb{R}^3$. Find a basis for $\mathbb{R}^3$ that contains the set S. Final answer: $\{(1,0,2), (-1,0,-1), (0,1,0)\}$.
9. Let $S = \{X_1, X_2, \cdots, X_n\}$ be a set of vectors in a vector space V. Show that S is a basis for V if and only if every vector in V can be expressed in exactly one way as a linar combination of the vectors in S. „ $\Rightarrow$ " : Use Theorem 4.4.1. And for „ $\Leftarrow$ " : Show the linear independence of S using the uniqueness.

4.5 Row Space, Column Space, and Null Space

Definition 4.5.1

Let $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \in M_{m\times n}$.

The set of rows of A are:

$$\left.\begin{array}{lcl} X_1 & = & [a_{11} \quad a_{12} \quad \cdots \quad a_{1n}] \\ X_2 & = & [a_{21} \quad a_{22} \quad \cdots \quad a_{2n}] \\ \vdots & & \qquad\qquad \vdots \\ X_m & = & [a_{m1} \quad a_{m2} \quad \cdots \quad a_{mn}] \end{array}\right\} \in \mathbb{R}^n$$

These row vectors span a subspace of $\mathbb{R}^n$ which is called **the row space of** A. Moreover, **the row rank** of A = **dim(row space of** A**)**.

Similarly, the columns of A are:

$$\left. Y_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, Y_2 = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}, \cdots, Y_n = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} \right\} \in \mathbb{R}^m.$$

These column vectors span a subspace of $\mathbb{R}^m$ which is called **the column space of** A. Moreover, **the column rank** of A = **dim(column space of** A**)**.

Moreover, the solution space of the homogeneous system $AX = O$ (which is a subspace of $\mathbb{R}^n$) is called the **null space of** A.

Remark 4.5.1

Let A be any $m \times n$ matrix, then

1. the row rank of A = the column rank of A = the rank of A = the rank of A^T.
2. n = the nullity of A + the rank of A.
3. m = the nullity of A^T + the rank of A.

Theorem 4.5.1

If A and B are two $m \times n$ row equivalent matrices, then they have the same row space.

In Example 4.5.1, we illustrate how to find bases for the row and column spaces of a given matrix.

Example 4.5.1

Let $A = \begin{bmatrix} 1 & -2 & 0 & 3 & -4 \\ 3 & 2 & 8 & 1 & 4 \\ 2 & 3 & 7 & 2 & 3 \\ -1 & 2 & 0 & 4 & -3 \end{bmatrix}$.

1. find a basis for the row space of A,
2. find a basis for the column space of A,
3. find a basis for the row space that contains only rows of A,
4. find a basis for the column space that contains only columns of A.

Solution:

1. To find a basis for the row space of A, we have to find the r.r.e.f. of A, then the set of non-zero rows of the r.r.e.f. forms a basis for the row space.

$$\begin{bmatrix} 1 & -2 & 0 & 3 & -4 \\ 3 & 2 & 8 & 1 & 4 \\ 2 & 3 & 7 & 2 & 3 \\ -1 & 2 & 0 & 4 & -3 \end{bmatrix} \approx \underset{\cdots\cdots}{\text{r.r.e.f.}} \approx \begin{bmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} \leftarrow \\ \leftarrow \\ \leftarrow \\ \end{matrix}$$

Therefore, the set $\{(1,0,2,0,1),(0,1,1,0,1),(0,0,0,1,-1)\}$ forms a basis for the row space of A. Note that the row rank of $A = 3$. (That is, nullity of $A = 5$ - $3 = 2$).

2. To find a basis for the column space of A, we have to find a basis for the row space of A^T. Therefore,

$$A^T = \begin{bmatrix} 1 & 3 & 2 & -1 \\ -2 & 2 & 3 & 2 \\ 0 & 8 & 7 & 0 \\ 3 & 1 & 2 & 4 \\ -4 & 4 & 3 & -3 \end{bmatrix} \approx \underset{\cdots\cdots}{\text{r.r.e.f.}} \approx \begin{bmatrix} 1 & 0 & 0 & \frac{11}{24} \\ 0 & 1 & 0 & \frac{-49}{24} \\ 0 & 0 & 1 & \frac{7}{3} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} \leftarrow \\ \leftarrow \\ \leftarrow \\ \\ \end{matrix}$$

Therefore, the set $\left\{(1,0,0,\frac{11}{24}),(0,1,0,\frac{-49}{24}),(0,0,1,\frac{7}{3})\right\}$ is a basis for the row space of A^T and it is a basis for the column space of A. The column rank of $A = 3$. (That is, nullity of $A = 5$ - $3 = 2$).

3. To find a basis for the row space of A that contains only rows of A, we do as follows:

$$A^T = \begin{bmatrix} 1 & 3 & 2 & -1 \\ -2 & 2 & 3 & 2 \\ 0 & 8 & 7 & 0 \\ 3 & 1 & 2 & 4 \\ -4 & 4 & 3 & -3 \end{bmatrix} \approx \overset{\text{r.r.e.f.}}{\cdots\cdots} \approx \begin{bmatrix} 1 & 0 & 0 & \frac{11}{24} \\ 0 & 1 & 0 & \frac{-49}{24} \\ 0 & 0 & 1 & \frac{7}{3} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
$$\uparrow \quad \uparrow \quad \uparrow$$

Then, the leading entries are pointing to column 1, column 2, and column 3 in the r.r.e.f. of A^T which correspond to row 1, row 2, and row 3 in A. Thus,

$$\{(1,-2,0,3,-4),(3,2,8,1,4),(2,3,7,2,3)\}$$

forms a basis for the row space of A containing only rows of A.

4. To find a basis for the column space of A that only contains columns of A, we do the following:

$$\begin{bmatrix} 1 & -2 & 0 & 3 & -4 \\ 3 & 2 & 8 & 1 & 4 \\ 2 & 3 & 7 & 2 & 3 \\ -1 & 2 & 0 & 4 & -3 \end{bmatrix} \approx \overset{\text{r.r.e.f.}}{\cdots\cdots} \approx \begin{bmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$
$$\uparrow \quad \uparrow \quad \quad \uparrow$$

Then, the leading entries are pointing to column 1, column 2 , and column 3 in the r.r.e.f. of A . Thus,

$$\{(1,3,2,-1),(-2,2,3,3),(3,1,2,4)\}$$

forms a basis for the column space of A containing only columns of A.

⋆ **Rank and Singularity:**

Theorem 4.5.2: Equivalent Statements

If A is an $n \times n$ matrix, then the following statements are equivalent:

1. A is invertible.
2. $AX = O$ has only the trivial solution.
3. A is row equivalent to I_n.
4. $AX = B$ has a unique solution for every $n \times 1$ matrix B.
5. $\det(A) \neq 0$.
6. The column vectors of A are: linearly independent; span $\mathbb{R}^n$; and form a basis for $\mathbb{R}^n$.
7. The row vectors of A are: linearly independent; span $\mathbb{R}^n$; and form a basis for $\mathbb{R}^n$.
8. A has rank n.
9. A has nullity 0.

If A is an $m \times n$ matrix, then the **smallest** possible rank of A is 0 (when A is the zero matrix), while the **largest** possible rank of A:

1. n (if $m \geq n$): When every column of the r.r.e.f. of A contains a leading 1.
2. m (if $m < n$): When every column of the r.r.e.f. of A contains a leading 1.

Also: the **largest** nullity of A is n (when rank is 0) and the **smallest** nullity of A is:

1. 0 (if $m \geq n$): When every column of the r.r.e.f. of A contains a leading 1.
2. $n - m$ (if $m < n$): When every column of the r.r.e.f. of A contains a leading 1.

Let A be a 3×5 matrix. Then: the largest possible rank of A is 3 and the smallest possible rank of A is 0 (the zero matrix). This is because, rank of A = row rank = column rank, and we only have 3 rows. Also, the largest nullity of A is 5 (zero matrix) and the smallest nullity is 2 (when rank of $A = 3$). Moreover, the largest possible rank of A^T is 3, and the largest possible nullity of A^T is 3.

Example 4.5.2

If A is a 5×3 matrix, show that A has linearly dependent rows.

Solution:

The largest possible rank of A is 3 and thus A must has at least two linearly dependent rows.

Example 4.5.3

Show that $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$ has rank 2 iff at least one of $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}, \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$ is nonzero.

Solution:

We have: rank of A is 2 iff column rank of A is 2 iff basis of column space has two columns iff two columns are linearly independent iff one of the determinants is nonzero.

Example 4.5.4

Let $A = \begin{bmatrix} 1 & 1 & 4 & 1 & 2 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 1 & -1 & 0 & 0 & 2 \\ 2 & 1 & 6 & 0 & 1 \end{bmatrix}$. Find the rank of A and the nullity of A.

Solution:

To find the rank and the nullity of A, we find any basis of any kind of A. So,

$$\begin{bmatrix} 1 & 1 & 4 & 1 & 2 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 1 & -1 & 0 & 0 & 2 \\ 2 & 1 & 6 & 0 & 1 \end{bmatrix} \approx \overset{\text{r.r.e.f.}}{\cdots\cdots} \approx \begin{bmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & 2 & 0 & -1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} \leftarrow \\ \leftarrow \\ \leftarrow \\ \\ \\ \end{matrix}$$

Therefore $\{(1, 0, 2, 0, 1), (0, 1, 2, 0, -1), (0, 0, 0, 1, 2)\}$ is a basis for the row space of A and rank(A) $= 3$ which implies that nullity of $A = 5 - 3 = 2$.

Example 4.5.5

Let $A = \begin{bmatrix} 1 & -2 & 0 & 3 & -4 \\ 3 & 2 & 8 & 1 & 4 \\ 2 & 3 & 7 & 2 & 3 \\ -1 & 2 & 0 & 4 & -3 \end{bmatrix}$.

1. Find bases for the row and column spaces of A,
2. Find a basis for the null space of A. Find nullity of A and nullity of A^T.
3. Does $X = (1, 2, 4, 3, 0)$ belong to the row space of A? Explain.
4. Express each column of A not in the basis of column space as a linear combination of the vectors in the basis you got in step 1.

Solution:

1. To get bases for the row space and column spaces of A, we do the following:

$$\begin{bmatrix} 1 & -2 & 0 & 3 & -4 \\ 3 & 2 & 8 & 1 & 4 \\ 2 & 3 & 7 & 2 & 3 \\ -1 & 2 & 0 & 4 & -3 \end{bmatrix} \approx \overset{\text{r.r.e.f.}}{\cdots\cdots} \approx \begin{array}{c} \begin{bmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{l} \leftarrow \\ \leftarrow \\ \leftarrow \\ \\ \end{array} \\ \begin{array}{ccccc} \uparrow & \uparrow & & \uparrow & \end{array} \end{array}$$

Thus, the set $\{(1, 0, 2, 0, 1), (0, 1, 1, 0, 1), (0, 0, 0, 1, -1)\}$ forms a basis for the row space of A, while the set $\{(1, 3, 2, -1), (-2, 2, 3, 2), (3, 1, 2, 4)\}$ forms a basis for the column space of A that only contains columns of A, but this is fine since there is no restrictions on the basis of column space of A mentioned in the question.

2. Using what we got in the previous step, the solution space of the homogeneous system is:

$$\begin{aligned} x_1 + 2x_3 + x_5 &= 0 \\ x_2 + x_3 + x_5 &= 0 \\ x_4 - x_5 &= 0 \end{aligned}$$

Let $x_5 = t, x_3 = r$, where $t, r \in \mathbb{R}$ to get

$$X = \begin{bmatrix} -2r - t \\ -r - t \\ r \\ t \\ t \end{bmatrix} = r \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$

Therefore, $\{(-2,-1,1,0,0),(-1,-1,0,1,1)\}$ is a basis for the null space of A. The nullity of A is 2 while the rank$(A) = 3$. Also, nullity of $A^T = 4\,$(number of rows in $A) - 3 = 1$.

3. Yes. It is clear that $X = (1)(1,0,2,0,1) + (2)(0,1,1,0,1) + (3)(0,0,0,1,-1)$ where those vectors are the vectors of the basis of the row space that were found in (1). It is also possible to consider the non-homogenous system $X = c_1\,(1,0,2,0,1) + c_2\,(0,1,1,0,1) + c_3\,(0,0,0,1,-1)$ to find the same answer.
4. Let the columns of A called $X_1, \cdots, X_5$. Then, we will express X_3 and X_5 (not in the basis) as a linear combination of the vectors (those in the basis) $\{X_1, X_2, X_4\}$. We can do so by looking at the r.r.e.f. form we got in step 1. For X_3: The third column of the rref matrix suggest that $X_3 = 2\,X_1 + X_2 + 0\,X_4$. For X_5: The fifth column of the rref matrix suggest that $X_5 = X_1 + X_2 - X_4$. Can you confirm that!?

Exercise 4.5.1

1. Let $A = \begin{bmatrix} -1 & -1 & 0 \\ 2 & 0 & 4 \end{bmatrix}$. Find a basis for the null space of A and determine the nullity of A. Final answer: $S = \{X_1 = (1,2,3), X_2 = (0,1,1), X_3 = (1,0,0)\}$

2. Let $A = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

   (a) Find rank(A), nullity(A), rank(A^T), and nullity(A^T).

   (b) Find a basis for the null space of A.

   (c) Find a basis for the row space of A^T.

   (d) Find a basis for the row space of A.

   Final answer:

   (a) rank$(A) = 2$, nullity$(A) = 2$, rank$(A^T) = 2$, and nullity$(A^T) = 1$.

   (b) a basis for the null space of $A = \{(1,0,0,0), (0,0,1,0)\}$.

   (c) a basis for the row space of $A^T = \{(0,1,0), (-1,0,1)\}$.

   (d) a basis for the row space of $A = \{(0,1,0,0), (0,0,0,1)\}$.

3. Let $A = \begin{bmatrix} 1 & 0 & -1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix}$.

   (a) Find a basis for the null space of A.

   (b) Find a basis for the row space of A^T.

   (c) Find a basis for the column space of A.

   Final answer:

   (a) rank$(A) = 2$, nullity$(A) = 2$, rank$(A^T) = 2$, and nullity$(A^T) = 1$.

   (b) a basis for the null space of $A = \{(1,0,0,0), (0,0,1,0)\}$.

   (c) a basis for the row space of $A^T = \{(0,1,0), (-1,0,1)\}$.

   (d) a basis for the row space of $A = \{(0,1,0,0), (0,0,0,1)\}$.

4. Let $S = \{X_1, X_2, X_3, X_4, X_5\}$, where $X_1 = (1,-2,0,3), X_2 = (2,-5,-3,6), X_3 = (0,1,3,0), X_4 = (2,-1,4,-7)$, and $X_5 = (5,-8,1,2)$.

   (a) Find a subset of S that forms a basis for the subspace **span** S.

(b) Express each vector not in the basis as a linear combination of the basis vectors.

(c) If A is the 4×5 matrix whose columns are the vectors of S in order, then find a basis for the row space of A, a basis for the column space of A, and a basis for the null space of A. Further, what is the nullity of A and the nullity of A^T.

Chapter

5 Eigenvalues and Eigenvectors

5.1 Eigenvalues and Eigenvectors

Definition 5.1.1

Let $A \in M_{n\times n}$. The real number λ is called an **eigenvalue** of A if there exists a nonzero vector $X \in \mathbb{R}^n$ so that

$$AX = \lambda X \qquad X \neq \mathbf{0}.$$

In this case, X is called an **eigenvector corresponding to** λ. This is called the eigenproblem.

For example, let $A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ and $X = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Then,

$$AX = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 2X.$$

Therefore, $\lambda = 2$ is an eigenvalue of A corresponding to eigenvector X.

Definition 5.1.2

If A is an $n \times n$ matrix, then $p_A(\lambda) = |\lambda I_n - A|$ is called the **characteristic polynomial** of A.

Theorem 5.1.1

If A is an $n \times n$ matrix, then λ is an eigenvalue of A iff $p_A(\lambda) = |\,(\lambda I_n - A)\,| = 0$.

Proof:

λ is an eigenvalue of A **iff** λ satisfies $AX = \lambda X$, with $X \neq O$ **iff** λ satisfies $\lambda X - AX = O$, with $X \neq O$ **iff** $(\lambda I_n - A)X = O$ has a nontrivial solutions **iff** $|\,\lambda\, I_n - A\,| = 0$.

Example 5.1.1

Let $A = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$. Find the eigenvalues of A.

Solution:

We first compute the characteristic polynomial $p_A(\lambda) = |\lambda I_3 - A| = 0$ as follows:

$$\begin{vmatrix} \lambda - 1 & 0 & -2 \\ -1 & \lambda & 0 \\ 0 & 0 & \lambda + 1 \end{vmatrix} = \lambda \begin{vmatrix} \lambda - 1 & -2 \\ 0 & \lambda + 1 \end{vmatrix} = \lambda(\lambda - 1)(\lambda + 1) = 0.$$

which implies that

$$\boxed{\lambda_1 = -1}, \ \boxed{\lambda_2 = 0}, \ \boxed{\lambda_3 = 1}.$$

Theorem 5.1.2

Let A be an $n \times n$ matrix. Then

1. The $\left| A \right|$ is the product of the eigenvalues of A.
2. A is invertible if and only if $\lambda = 0$ is not an eigenvalue of A.

Proof:

1. Assume that $\lambda_1, \lambda_2, \cdots, \lambda_n$ are the eigenvalues of A. Then,

$$\begin{aligned} p_A(\lambda) = |\lambda I_n - A| &= (\lambda - \lambda_1)(\lambda - \lambda_2)\cdots(\lambda - \lambda_n), \\ \text{setting } \lambda = 0,\ p_A(0) = |-A| &= (-\lambda_1)(-\lambda_2)\cdots(-\lambda_n), \\ (-1)^n \left| A \right| &= (-1)^n\, \lambda_1\lambda_2\cdots\lambda_n, \\ \left| A \right| &= \lambda_1\lambda_2\cdots\lambda_n. \end{aligned}$$

2. A is invertible **iff** $\left| A \right| \neq 0$ **iff** $\left| A \right| = \lambda_1\lambda_2\cdots\lambda_n \neq 0$ **iff** $\lambda_i \neq 0$ for all $1 \leq i \leq n$.

Theorem 5.1.3

If A is an $n \times n$ triangular matrix (upper triangular, lower triangular, or diagonal), then the eigenvalues of A are the entries on the main diagonal.

Proof:

If $A = [a_{ij}]$ (where $1 \leq i, j \leq n$) is a triangular matrix, then $\lambda I_n - A$ is also a triangular matrix, and its main diagonal entries are $[\lambda_i - a_{ii}]$ for $1 \leq i \leq n$. Recall that the determinant of a triangular matrix is the product of its main diagonal entries. Thus,

$$p_A(\lambda) = |\lambda I_n - A| = 0 \quad \Rightarrow \quad (\lambda - a_{11})(\lambda - a_{22}) \cdots (\lambda - a_{nn}) = 0.$$

Therefore, $\lambda_1 = a_{11}$, $\lambda_2 = a_{22}$, $\cdots$, $\lambda_n = a_{nn}$.

Theorem 5.1.4

If A is an $n \times n$ matrix, then: λ is an eigenvalue of A **iff** The system $(\lambda I_n - A)X = O$ has nontrivial solutions **iff** There is a nonzero vector X such that $AX = \lambda X$ **iff** λ is a solution of $p_A(\lambda) = |\,\lambda I_n - A\,| = 0$

Definition 5.1.3

Let A be an $n \times n$ matrix with an eigenvalue λ. The **eigenspace** of A corresponding to λ, denoted $\mathbf{E}_\lambda$, is defined as the solution space of the homogeneous system $(\lambda I_n - A)X = O$. That is, $\mathbf{E}_\lambda$ is the null space of the matrix $\lambda I_n - A$.

Example 5.1.2

Let $A = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$. Find bases for the eigenspaces of A. **OR**: Find the eigenvalues of A and the corresponding eigenvectors.

Solution:

As in Example 5.1.1, the characteristic polynomial $p_A(\lambda) = |\lambda I_3 - A| = 0$ as follows:

$$\begin{vmatrix} \lambda-1 & 0 & -2 \\ -1 & \lambda & 0 \\ 0 & 0 & \lambda+1 \end{vmatrix} = \lambda \begin{vmatrix} \lambda-1 & -2 \\ 0 & \lambda+1 \end{vmatrix} = \lambda(\lambda-1)(\lambda+1) = 0.$$

which implies that

$$\boxed{\lambda_1 = -1}, \quad \boxed{\lambda_2 = 0}, \quad \boxed{\lambda_3 = 1}.$$

Thus, there are three eigenspaces of A corresponding to these eigenvalues. To find bases for these eigenspaces, we solve the homogeneous system $(\lambda I_3 - A)X = O$, for $\lambda_1, \lambda_2, \lambda_3$. That is:

$$\left[\begin{array}{ccc|c} \lambda_i - 1 & 0 & -2 & 0 \\ -1 & \lambda_i & 0 & 0 \\ 0 & 0 & \lambda_i + 1 & 0 \end{array}\right]. \tag{5.1.1}$$

1. $\boxed{\lambda_1 = -1}$ $\Rightarrow$ $(\lambda_1 I_3 - A)X_1 = \mathbf{0}$, $X_1 = (a,b,c) \neq (0,0,0)$. Substitute $\lambda_1 = -1$ in Equation 5.1.1 to get:

$$\left[\begin{array}{ccc|c} -2 & 0 & -2 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \to \cdots \to \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Thus, $a + c = 0$, and $-b + c = 0$. That is, $a = -c$, and $b = c$. Let $c = t \in \mathbb{R}\backslash\{0\}$ to get $X_1 = \begin{bmatrix} -t \\ t \\ t \end{bmatrix}$. Choosing $t = 1$, we get a basis for $\mathbf{E}_{\lambda_1}$ containing the vector

$$P_1 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}.$$

2. $\boxed{\lambda_2 = 0}$ $\Rightarrow$ $(\lambda_2 I_3 - A)X_2 = \mathbf{0}$, $X_2 = (a,b,c) \neq (0,0,0)$. Substitute $\lambda_2 = 0$ in Equation 5.1.1 to get:

$$\left[\begin{array}{ccc|c} -1 & 0 & -2 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \to \cdots \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Thus, $a = c = 0$. Let $b = t \in \mathbb{R}\backslash\{0\}$ to get $X_2 = \begin{bmatrix} 0 \\ t \\ 0 \end{bmatrix}$. Choosing $t = 1$, we get a basis for

$\mathbf{E}_{\lambda_2}$ containing the vector

$$P_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

3. $\boxed{\lambda_3 = 1} \Rightarrow (\lambda_3 I_3 - A)X_3 = \mathbf{0},\ X_3 = (a, b, c) \neq (0, 0, 0)$. Substitute $\lambda_3 = 1$ in Equation 5.1.1 to get:

$$\left[\begin{array}{ccc|c} 0 & 0 & -2 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right] \to \cdots \to \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Thus, $a - b = 0$, and $c = 0$. If $b = t \in \mathbb{R}\backslash\{0\}$, then $a = t$ as well and we get $X_3 = \begin{bmatrix} t \\ t \\ 0 \end{bmatrix}$.

Choosing $t = 1$, we get a basis for $\mathbf{E}_{\lambda_3}$ containing the vector

$$P_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

Theorem 5.1.5

If k is a positive integer, λ is an eigenvalue of a matrix A, and X is a corresponding eigenvector, then λ^k is an eigenvalue of A^k and X is a corresponding eigenvector.

Proof:

If $AX = \lambda X$, then we have $A^2X = A(AX) = A(\lambda X) = \lambda(AX) = \lambda^2 X$. Applying this simple idea k times, we get

$$A^kX = A^{k-1}(AX) = \lambda\left(A^{k-1}X\right) = \lambda^2\left(A^{k-2}X\right) = \cdots = \lambda^k X.$$

Theorem 5.1.6

If λ is an eigenvalue of an invertible matrix A, and X is a corresponding eigenvector, then $\frac{1}{\lambda}$ is an eigenvalue of A^{-1} and X is a corresponding eigenvector.

Proof:

If $AX = \lambda X$ and A is invertible, then multiplying with A^{-1} both sides (from left), we get

$$A^{-1} \cdot AX = A^{-1} \cdot \lambda X \quad \to \quad X = \lambda\, A^{-1} X \quad \to \quad \frac{1}{\lambda} X = A^{-1} X.$$

Exercise 5.1.1

1. Show that A and A^T have the same eigenvalues. Hint: $|\lambda I_n - A| = \left|(\lambda I_n - A)^T\right|$.
2. Suppose that $p_A(x) = \lambda^2 (\lambda+3)^3 (\lambda - 4)$ is the characteristic polynomial of some matrix A. Then,
   (a) What is the size of A? Explain.
   (b) Is A invertible? Why?
   (c) How many eigenspaces does A have? Explain.
3. Find the eigenvalues and bases for the eigenspaces of $A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & -1 & -4 \\ -1 & -1 & 2 \end{bmatrix}$ Final result: $\lambda_1 = -2, \lambda_2 = 1, \lambda_3 = 3$. And $P_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$, $P_2 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$, and $P_3 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$.
4. Find the eigenvalues of $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ k^3 & -3k^2 & 3k \end{bmatrix}$ Final result: $\lambda = k$ since $p(\lambda) = (\lambda - k)^3$.
5. Show that if a, b, c, d are integers such that $a + b = c + d$, then $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ has integer eigenvalues $\lambda_1 = a + b$ and $\lambda_2 = a - c$. Hint: Use your algebraic abilities.

5.2 Diagonalization

Definition 5.2.1

A matrix B is said to be **similar** to matrix A, denoted by $B \equiv A$, if there exists a non-singular matrix P such that $B = P^{-1}AP$.

⋆ **Properties of Similar Matrices:**

1. $A \equiv A$ since $A = I^{-1}AI$.
2. if $B \equiv A$, then $A \equiv B$.

   Proof. If $B \equiv A$, then $\exists\, P, P^{-1}$ such that $B = P^{-1}AP$ or $PBP^{-1} = A$. Let $Q = P^{-1}$ to get $A = Q^{-1}BQ$. Thus, $A \equiv B$. □

3. if $A \equiv B$ and $B \equiv C$, then $A \equiv C$.

   Proof.

$$\begin{aligned} A \equiv B \quad &\Rightarrow \quad \exists\; P, P^{-1} \text{ such that } A = P^{-1}BP, \\ B \equiv C \quad &\Rightarrow \quad \exists\; Q, Q^{-1} \text{ such that } B = Q^{-1}CQ. \end{aligned}$$

   Therefore,

$$A = P^{-1}BP = P^{-1}Q^{-1}C\,QP = (QP)^{-1}\,C\,(QP) \;\Rightarrow\; A \equiv C.$$

   □

4. if $A \equiv B$, then $\left|\,A\,\right| = \left|\,B\,\right|$.

   Proof. If $A \equiv B$, then there exists P, P^{-1} such that $B = P^{-1}\,A\,P$ with $|P| \neq 0$. Therefore,

$$\left|\,B\,\right| = \left|P^{-1}\,A\,P\right| = |P^{-1}|\left|\,A\,\right||P| = \frac{1}{\cancel{|P|}}\left|\,A\,\right|\cancel{|P|} = \left|\,A\,\right|.$$

   □

5. if $A \equiv B$, then $A^T \equiv B^T$.

Proof. If $A \equiv B$, then there exist P, P^{-1} such that $B = P^{-1}\,A\,P$. Thus,

$$\begin{aligned} B &= P^{-1}\,A\,P, \\ B^T &= (P^{-1}\,A\,P)^T, \\ B^T &= P^T\,A^T\,(P^{-1})^T, \\ B^T &= P^T\,A^T\,(P^T)^{-1}. \end{aligned}$$

Let $Q^{-1} = P^T$, to get $B^T = Q^{-1}\,A^T\,Q$. Therefore, $B^T \equiv A^T$. □

Theorem 5.2.1

Similar matrices have the same eigenvalues.

Proof:

Let A and B be two similar $n \times n$ matrices. Then, there is an invertible matrix P such that $B = P^{-1}\,A\,P$. Then,

$$\begin{aligned} p_B(\lambda) = |\lambda I_n - B| = \left|\lambda I_n - P^{-1}\,A\,P\right| = \left|P^{-1}(\lambda P\,P^{-1} - A)P\right| \\ = \cancel{|P^{-1}|}\,|\lambda I_n - A|\,\cancel{|P|} = |\lambda I_n - A| = p_A(\lambda). \end{aligned}$$

The characteristic polynomials of A and B are the same. Hence they have the same eigenvalues.

Definition 5.2.2

An $n \times n$ matrix A is **diagonalizable** if and only if A is similar to a diagonal matrix D, i.e.

$$D = P^{-1}\,A\,P \quad \text{with} \quad |P| \neq 0.$$

D: its diagonal entries are the eigenvalues of A. That is: $D = diag(\lambda_1, \lambda_2, \cdots, \lambda_n)$.

P: its columns are the linearly independent eigenvectors of A. That is $P = [P_1\,|\,P_2\,|\,\cdots\,|\,P_n]$.

Theorem 5.2.2

A matrix A has linearly independent eigenvectors if all of its eigenvalues are real and distinct.

Theorem 5.2.3

An $n \times n$ matrix A is diagonalizable if and only if it has n linearly independent eigenvectors.

Definition 5.2.3

If A is an $n \times n$ matrix with eigenvalue λ_0, then the number of times that $\lambda - \lambda_0$ appears as a factor in the characteristic polynomial of A is called the **algebraic multiplicity** of λ_0.

Theorem 5.2.4

Let A be a square matrix. Then A is diagonalizable iff every eigenspace of A corresponding to eigenvalue λ_i has its dimension equals to the algebraic multiplicity of λ_i.

Example 5.2.1

Let $A = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$. If possible, find matrices P and D so that A is diagonalizable.

Solution:

Recall that in Example 5.1.1, we found $\lambda_1 = -1, \lambda_2 = 0$, and $\lambda_3 = 1$ with bases

$$P_1 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, P_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, P_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

Since, we have real and distinct eigenvalues, the eigenvectors P_1, P_2, and P_3 are linearly independent. Thus, A is diagonalizable and

$$D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } P = \begin{bmatrix} -1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

Example 5.2.2

Find a matrix P that diagonalizes

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}.$$

Solution:

We first compute the characteristic polynomial $p_A(\lambda) = |\lambda I_3 - A| = 0$ as follows:

$$\begin{vmatrix} \lambda & 0 & -1 \\ 0 & \lambda - 1 & -2 \\ 0 & 0 & \lambda - 1 \end{vmatrix} = \lambda(\lambda - 1)(\lambda - 1) = 0$$

Thus, $\lambda_1 = 1$, $\lambda_2 = 1$, and $\lambda_3 = 0$. To find the corresponding bases for eigenspaces of A, we solve the homogeneous system

$$\left[\begin{array}{ccc|c} \lambda_i & 0 & -1 & 0 \\ 0 & \lambda_i - 1 & -2 & 0 \\ 0 & 0 & \lambda_i - 1 & 0 \end{array}\right]. \tag{5.2.1}$$

1. $\boxed{\lambda_1 = \lambda_2 = 1}$ $\Rightarrow$ $(\lambda_1 I_3 - A)X_1 = \mathbf{0}$, $X_1 = (a, b, c) \neq (0, 0, 0)$. Substitute $\lambda_1 = \lambda_2 = 1$ in Equation (5.2.1) to get:

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Thus, we get $a - c = 0$ and $-2c = 0$ which implies that $a = c = 0$. If $b = t \in \mathbb{R} \backslash \{0\}$, then we get $X_1 = \begin{bmatrix} 0 \\ t \\ 0 \end{bmatrix}$. We choose $t = 1$ to get a basis for $\mathbf{E}_{\lambda_1}$ with one vector $P_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

We see here that the dimension of $\mathbf{E}_{\lambda_1}$ is 1 while the multiplicity of λ_1 is 2. Therefore, A is not diagonalizable.

Example 5.2.3

Find, if possible, matrices D and P so that $D = P^{-1}\,A\,P$, where

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}.$$

Solution:

We first compute the characteristic polynomial $p_A(\lambda) = |\lambda I_3 - A| = 0$ as follows:

$$\begin{vmatrix} \lambda - 1 & 0 & 0 \\ 0 & \lambda & 0 \\ 1 & 0 & \lambda \end{vmatrix} = \lambda^2(\lambda - 1) = 0.$$

Thus, $\lambda_1 = 0$, $\lambda_2 = 0$, and $\lambda_3 = 1$. To find the corresponding bases for eigenspaces of A, we solve the homogeneous system

$$\left[\begin{array}{ccc|c} \lambda_i - 1 & 0 & 0 & 0 \\ 0 & \lambda_i & 0 & 0 \\ 1 & 0 & \lambda_i & 0 \end{array}\right]. \tag{5.2.2}$$

1. $\boxed{\lambda_1 = \lambda_2 = 0}$ $\Rightarrow$ $(\lambda_1 I_3 - A)X_1 = \mathbf{0}$, $X_1 = (a, b, c) \neq (0, 0, 0)$. Substitute $\lambda_1 = \lambda_2 = 0$ in Equation 5.2.2 to get:

$$\left[\begin{array}{ccc|c} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right].$$

   Thus, we get $a = 0$. If $b = t$ and $c = r$ (not both zeros) be two real numbers, we get

$$X_1 = \begin{bmatrix} 0 \\ t \\ r \end{bmatrix} = t \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + r \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \text{if} \begin{Bmatrix} t = 1 \\ r = 0 \end{Bmatrix} \Rightarrow P_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \text{if} \begin{Bmatrix} t = 0 \\ r = 1 \end{Bmatrix} \Rightarrow P_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

   Here, the dimension of $\mathbf{E}_{\lambda_1}$ is 2 which equals to the algebraic multiplicity of $\lambda = 0$. So, we continue with the other eigenvalues.

2. $\boxed{\lambda_3 = 1}$ $\Rightarrow$ $(\lambda_3 I_3 - A)X_3 = \mathbf{0}$, $X_3 = (a, b, c) \neq (0, 0, 0)$. Substitute $\lambda_3 = 1$ in Equation 5.2.2 to get:

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right].$$

Thus, we get $b = 0$ and $a + c = 0$. If $c = t \in \mathbb{R}\backslash\{0\}$, we get $X_3 = \begin{bmatrix} t \\ 0 \\ -t \end{bmatrix}$. We choose $t = 1$ to get a basis with one vector $P_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$.

There are three basis vectors in total, so the matrix $P = [P_1 \,|\, P_2 \,|\, P_3]$ diagonalize A and we get $D = P^{-1}AP = diag(1, 0, 0)$.

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}.$$

Exercise 5.2.1

1. Show that similar matrices have the same trace. Hint: Recall that $tr(AB) = tr(BA)$.
2. Show that A and B in each of the following are not similar matrices:
   (a) $A = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 \\ 3 & 3 \end{bmatrix}$.
   (b) $A = \begin{bmatrix} 4 & 2 & 0 \\ 2 & 1 & 0 \\ 1 & 1 & 7 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 3 & 4 \\ 0 & 7 & 2 \\ 0 & 0 & 4 \end{bmatrix}$.

   Hint: Similar matrices shares some properties like determinants and traces.
3. Let $A = \begin{bmatrix} 0 & -3 & -3 \\ 1 & 4 & 1 \\ -1 & -1 & 2 \end{bmatrix}$.
   (a) Find the eigenvalues of A.
   (b) For each eigenvalue λ, find the rank of the matrix $\lambda\, I_3 - A$.
   (c) Is A diagonalizable? Why?

   Hint: For part 3, use what you got in part 2 and recall that for $n \times n$ matrix, we have $n = rank - nullity$.
4. Show that if A is diagonalizable, then
   (a) A^T is diagonalizable.
   (b) A^k is diagonalizable, for any positive integer k.

   Hint: A is diagonalizable implies that $A = P\,D\,P^{-1}$.
5. Let $A = \begin{bmatrix} 1 & -2 & 8 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.
   (a) Find A^{10000}.
   (b) Find A^{20021}.
   (c) Find A^{-20021}.

   Hint: Write A in the form $A = P\,D\,P^{-1}$.
6. Show that if A and B are invertible matrices, then AB and BA are similar. Hint: They are similar if $AB = (\star)^{-1}\,(BA)\,(\star)$.

7. Prove: If A and B are $n \times n$ invertible matrices, then $A\,B^{-1}$ and $B^{-1}\,A$ have the same eigenvalues. Hint: Show that they have the same characteristic polynomial.

8. Let $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 0 \\ 1 & a & 2 \end{bmatrix}$, where $a \in \mathbb{R}$.

   (a) Find all eigenvalues of A.

   (b) For $a = -2$, determine whether A is diagonalizable.

   (c) For $a \neq -2$, find all eigenvectors of A.

   Final answer: Eigenvalues: 1 and 2. If $a = -2$, A is diagonalizable. Otherwise, A is not diagonalizable.

9. (a) Show that a square matrix A is singular iff it has an eigenvalue 0.

   (b) Use part 1 to show that 0 is an eigenvalue of the matrix $A = \begin{bmatrix} 2017 & -2017 & 2020 \\ 2018 & -2018 & 2021 \\ 2019 & -2019 & 2022 \end{bmatrix}$.

3.5 The Cross Product in $\mathbb{R}^n$

Recall that if $0 \le \theta \le \pi$ is an angle between two vectors X and Y in $\mathbb{R}^n$, then

$$-1 \le \cos\theta = \frac{X \cdot Y}{\|X\|\,\|Y\|} \le 1 \qquad \text{or} \qquad X \cdot Y = \|X\|\,\|Y\|\,\cos\theta \tag{3.5.1}$$

Definition 3.5.1

Two nonzero vectors X and Y in $\mathbb{R}^n$ are said to be **orthogonal** (or **perpendicular**) if $X \cdot Y = 0$.

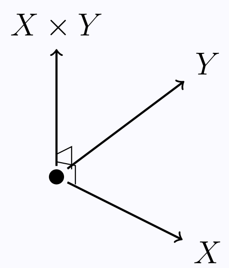
Remark 3.5.1

Let $X, Y \in \mathbb{R}^n$ and $0 \le \theta \le \pi$, then

1. $X \perp Y$ (**orthogonal**) $\Leftrightarrow \theta = \frac{\pi}{2} \Leftrightarrow \cos\theta = 0 \Leftrightarrow X \cdot Y = 0$.
2. $X \,//\, Y$ (**parallel** (same direction)) $\Leftrightarrow \theta = 0 \Leftrightarrow \cos\theta = 1 \Leftrightarrow X \cdot Y = \|X\|\,\|Y\| \Leftrightarrow Y = c\,X$ with $c > 0$.
3. $X \,//\, Y$ (parallel (opposite direction)) $\Leftrightarrow \theta = \pi \Leftrightarrow \cos\theta = -1 \Leftrightarrow X \cdot Y = -\|X\|\,\|Y\| \Leftrightarrow Y = c\,X$ with $c < 0$.

Definition 3.5.2

Let $X = x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}$ and $Y = y_1\mathbf{i} + y_2\mathbf{j} + y_3\mathbf{k}$ be two vectors in $\mathbb{R}^3$, then the **cross product** of X and Y, denoted by $X \times Y$, is defined by



$$X \times Y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = \begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} x_1 & x_3 \\ y_1 & y_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} \mathbf{k}$$

That is

$$X \times Y = (x_2\,y_3 - x_3\,y_2, x_1\,y_3 - x_3\,y_1, x_1\,y_2 - x_2\,y_1).$$

Example 3.5.1

Find $X \times Y$ where $X = 2\mathbf{i} + \mathbf{j} + 2\mathbf{k}$ and $Y = 3\mathbf{i} - \mathbf{j} - \mathbf{k}$.

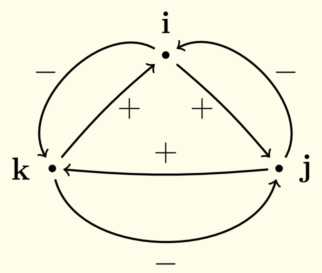
Solution:

$$X \times Y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 2 \\ 3 & -1 & -1 \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ -1 & -1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 2 & 2 \\ 3 & -1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 2 & 1 \\ 3 & -1 \end{vmatrix} \mathbf{k} = \mathbf{i} + 8\mathbf{j} - 5\mathbf{k} = (1, 8, -5).$$

Remark 3.5.2

It can be shown (Try it your self) that

$\mathbf{i} \times \mathbf{i} = \mathbf{0}$	$\mathbf{j} \times \mathbf{j} = \mathbf{0}$	$\mathbf{k} \times \mathbf{k} = \mathbf{0}$
$\mathbf{i} \times \mathbf{j} = \mathbf{k}$	$\mathbf{j} \times \mathbf{k} = \mathbf{i}$	$\mathbf{k} \times \mathbf{j} = -\mathbf{i}$
$\mathbf{i} \times \mathbf{k} = -\mathbf{j}$	$\mathbf{j} \times \mathbf{i} = -\mathbf{k}$	$\mathbf{k} \times \mathbf{i} = \mathbf{j}$



Theorem 3.5.1: Properties of Cross Product

Let $X, Y, Z \in \mathbb{R}^3$ and $c \in \mathbb{R}$. Then, (Can you prove the following propertes?)

1. $X \times Y = -(Y \times X)$,
2. $X \times (Y + Z) = X \times Y + X \times Z$,
3. $(X + Y) \times Z = X \times Z + Y \times Z$,
4. $cX \times Y = X \times cY = c(X \times Y)$,
5. $X \times X = O$,
6. $X \times O = O \times X = O$,
7. $X \cdot (X \times Y) = Y \cdot (X \times Y) = 0$, (**X × Y is orthogonal to X and Y**)
8. $\| X \times Y \|^2 = \| X \|^2 \| Y \|^2 - (X \cdot Y)^2$, (**Lagrange's identity**)
9. $X \times (Y \times Z) = (X \cdot Z)Y - (X \cdot Y)Z$, (**triple vector product**)
10. $(X \times Y) \times Z = (X \cdot Z)Y - (Y \cdot Z)X$. (**triple vector product**)

Example 3.5.2

Let $X = x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}$; $Y = y_1\mathbf{i} + y_2\mathbf{j} + y_3\mathbf{k}$; $Z = z_1\mathbf{i} + z_2\mathbf{j} + z_3\mathbf{k} \in \mathbb{R}^3$. Show that

$$(X \times Y) \cdot Z = X \cdot (Y \times Z) = \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix}.$$

Solution:

$$\begin{aligned} \text{L.H.S.} = (X \times Y) \cdot Z &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} \cdot (z_1\mathbf{i} + z_2\mathbf{j} + z_3\mathbf{k}) \\ &= \left(\begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} x_1 & x_3 \\ y_1 & y_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} \mathbf{k} \right) \cdot (z_1\mathbf{i} + z_2\mathbf{j} + z_3\mathbf{k}) \\ &= \begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix} z_1 - \begin{vmatrix} x_1 & x_3 \\ y_1 & y_3 \end{vmatrix} z_2 + \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} z_3 = \text{R.H.S.} \end{aligned}$$

The proof of $X \cdot (Y \times Z)$ is similar.

Example 3.5.3

Let X, Y, Z be in $\mathbb{R}^3$ such that $(X \times Y) \cdot Z = 6$. Find a) $X \cdot (Y \times Z)$, b) $2X \cdot (Y \times Z)$, c) $X \cdot (Z \times Y)$ and d) $X \times (Y \times 4X)$.

Solution:

1. $X \cdot (Y \times Z) = 6$,
2. $2X \cdot (Y \times Z) = 12$,
3. $X \cdot (Z \times Y) = -6$,
4. $X \cdot (Y \times 4X) = 0$.

Example 3.5.4

Find a vector of length 12 so that it is perpendicular to both

$$X = 2\mathbf{i} - \mathbf{j} - 2\mathbf{k} \text{ and } Y = 2\mathbf{i} + \mathbf{j}.$$

Solution:

The vector $X \times Y$ is always orthogonal to both X and Y. So, we compute that vector and make its length equals to 12.

$$X \times Y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & -2 \\ 2 & 1 & 0 \end{vmatrix} = 2\mathbf{i} - 4\mathbf{j} + 4\mathbf{k} \qquad \text{and} \qquad \|X \times Y\| = \sqrt{4 + 16 + 16} = 6.$$

Therefore, $\frac{1}{6}(X \times Y)$ is a unit vector and orthogonal to both X and Y; while $12\left(\frac{1}{6}(X \times Y)\right) = 2(X \times Y)$ is a vector of length 12 and orthogonal to both X and Y.

Theorem 3.5.2

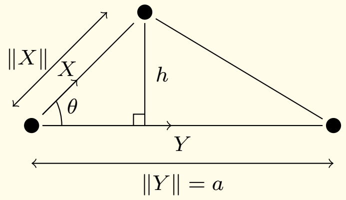
Let $X, Y \in \mathbb{R}^3$ have an angle θ between them. Then

$$\| X \times Y \| = \| X \| \, \| Y \| \sin\theta.$$

Let $X, Y, Z \in \mathbb{R}^3$, then

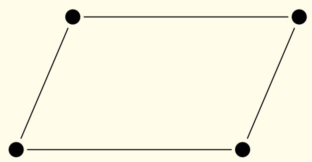
1. $X \perp Y \iff \theta = \frac{\pi}{2} \iff \sin\theta = 1 \iff \|X \times Y\| = \|X\|\|Y\|$,
2. $X // Y \iff \theta = 0 \text{ or } \pi \iff \|X \times Y\| = 0 \iff X \times Y = \mathbf{0}$,
3. Area of triangle:

$$\begin{aligned} A_\Delta &= \frac{1}{2}ah \\ \sin\theta &= \frac{h}{\|x\|} \\ \implies & \quad h = \|x\| \sin\theta \text{ and if } \|Y\| = a, \\ A_\Delta &= \frac{1}{2}\|Y\|\|X\|\sin\theta = \frac{1}{2}\|X \times Y\|. \end{aligned}$$



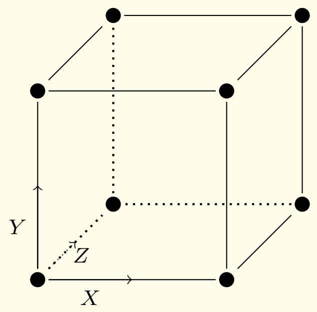
4. Area of parallel gram (two triangles):

$$A_{\square} = \|X \times Y\|.$$



5. Volume of parallel piped:

$$Volume = |X \cdot (Y \times Z)|.$$



Example 3.5.5

Find the area of the triangle with vertices: $P_1(2, 2, 4)$, $P_2(-1, 0, 5)$, and $P_3(3, 4, 3)$.

Solution:

Let $X = \overrightarrow{P_1P_2} = P_2 - P_1 = (-3, -2, 1)$ and $Y = \overrightarrow{P_1P_3} = P_3 - P_1 = (1, 2, -1)$. Then,

$$\begin{aligned} X \times Y &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & -2 & 1 \\ 1 & 2 & -1 \end{vmatrix} \\ &= \begin{vmatrix} -2 & 1 \\ 2 & -1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} -3 & 1 \\ 1 & -1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} -3 & -2 \\ 1 & 2 \end{vmatrix} \mathbf{k} = 0\mathbf{i} - 2\mathbf{j} - 4\mathbf{k}. \end{aligned}$$

Therefore, $\|X \times Y\| = \sqrt{4 + 16} = \sqrt{20} = 2\sqrt{5}$ and $A_{\Delta} = \frac{1}{2}\|X \times Y\| = \sqrt{5}$.

Example 3.5.6

Find the volume of the parallel piped with a vertex at the origin and edges $X = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$, $Y = \mathbf{i} + 3\mathbf{j} + \mathbf{k}$, and $Z = 2\mathbf{i} + \mathbf{j} + 2\mathbf{k}$.

Solution:

$$\begin{aligned}
\text{Volume} &= \big|X\cdot(Y\times Z)\big| = \big|(X\times Y)\cdot Z\big| = \left|\begin{vmatrix} \mathbf{1} & -2 & 3\\ \mathbf{1} & 3 & 1\\ \mathbf{2} & 1 & 2\end{vmatrix}\right| \\
&= \big|\mathbf{1}(6-1)-\mathbf{1}(-4-3)+\mathbf{2}(-2-9)\big| = \big|5+7-22\big| = |-10| = 10.
\end{aligned}$$

Exercise 3.5.1

Show that two nonzero vectors X and Y in $\mathbb{R}^3$ are parallel, if and only if, $X \times Y = 0$.

Solution:

$X//Y$ iff $Y = c\,X$ iff $X \times Y = X \times (cX) = O$.

Exercise 3.5.2

If U and V are nonzero vectors in $\mathbb{R}^3$ such that $\|(2U) \times (2V)\| = -4U \cdot V$, compute the angle between U and V.

Hint: What is θ if $\tan(\theta) = -1$.

Exercise 3.5.3

Find the area of the triangle whose vertices are $P(1, 0, -1)$, $Q(2, -1, 3)$ and $R(0, 1, -2)$.

Exercise 3.5.4

Let U and V be unit vectors in $\mathbb{R}^3$. Show that $\|U \times V\|^2 + (U \cdot V)^2 = 1$.

Exercise 3.5.5

Let X and Y be two nonzero vectors in $\mathbb{R}^3$, with angle $\theta = \frac{\pi}{3}$ between them. Find $\|\, X \times Y \,\|$, if $\|\, X \,\| = 3$ and $\|\, -2Y \,\| = 4$.

Exercise 3.5.6

If X and Y are two vectors in $\mathbb{R}^3$, show that $X \times Y$ is orthogonal to X.

Exercise 3.5.7

Find a vector that is orthogonal to both vectors $X = (0, 2, -2)$ and $Y = (1, 3, 0)$.

Exercise 3.5.8

Find the are of the parallelogram determined by $X = (1, 3, 4)$ and $Y = (5, 1, 2)$.

Final answer: $2\sqrt{131}$.

3.3 Orthogonality

Definition 3.3.1

Two nonzero vectors X and Y in $\mathbb{R}^n$ are said to be **orthogonal** (or **perpendicular**) if $X \cdot Y = 0$. A nonemoty set of vectors in $\mathbb{R}^n$ is called an **orthogonal set** if all pairs of distinct vectors in the set are orthogonal. An orthogonal set of unit vectors is called an **orthonormal set**.

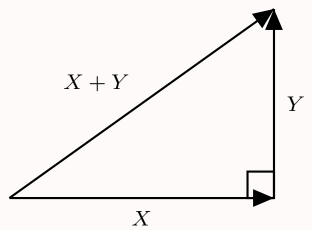
The set of standard unit vectors $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ in $\mathbb{R}^3$ is an orthonormal set.

The set $\{(1,-2,0),(2,1,2),(4,2,-5)\}$ is an orthogonal set since the dot product of any pair of distinct vectors is 0.

Theorem 3.3.1: The Pythagoras Theorem in $\mathbb{R}^n$

If X and Y are orthogonal vectors in $\mathbb{R}^n$, then

$$\| X+Y \|^2 = \| X \|^2 + \| Y \|^2$$



Proof:

Since X and Y are orthogonal, we have $X \cdot Y = 0$. Then

$$\| X+Y \|^2 = (X+Y)\cdot(X+Y) = \| X \|^2 + 2\,(X \cdot Y) + \| Y \|^2 = \| X \|^2 + \| Y \|^2.$$

Example 3.3.1

Find all vectors in $\mathbb{R}^4$ that are orthogonal to both

$$X = (1,1,1,1) \quad \text{and} \quad Y = (1,1,-1,-1).$$

Solution:

Let $Z = (a, b, c, d) \in \mathbb{R}^4$ so that $Z \cdot X = Z \cdot Y = 0$. Therefore, we get the following homogenous system:

$$\begin{aligned} a+b+c+d &= 0 \\ a+b-c-d &= 0 \end{aligned} \quad \Longrightarrow \quad \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & -1 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right]$$

Solving this system, we get $a = -b$ and $c = -d$. Let $b = r$ and $d = s$ where $r, s \in \mathbb{R}$ to get $Z = (-r, r, -s, s)$ which is the form of any vector in $\mathbb{R}^4$ that is orthogonal to both X and Y.

Example 3.3.2

Show that the triangle with vertices $P_1(2, 3, -4), P_2(3, 1, 2)$, and $P_3(7, 0, 1)$ is a right triangle.

Solution:

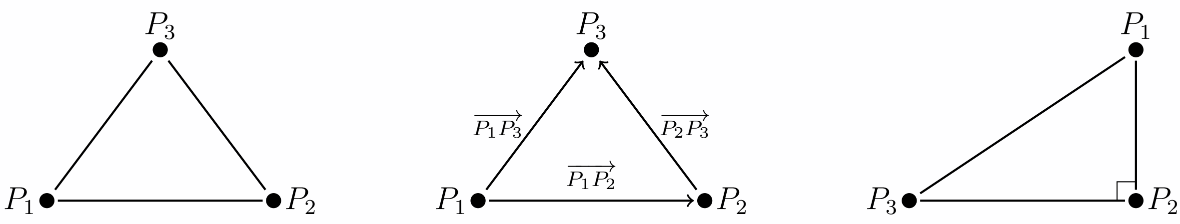


Figure 1 Figure 2 Figure 3

We start with Figure 1 as we do not know if there is a right angle. We create three vectors, namely

$$\begin{array}{ccccccc} X & = & \overrightarrow{P_1P_2} & = & P_2 - P_1 & = & (1, -2, 6) \\ Y & = & \overrightarrow{P_1P_3} & = & P_3 - P_1 & = & (5, -3, 5) \\ Z & = & \overrightarrow{P_2P_3} & = & P_3 - P_2 & = & (4, -1, -1) \end{array}$$

This is draw in Figure 2. Then, we want to find two vectors whose dot product is zero which is valid by considering X and Z. That is

$$X \cdot Z = 4 + 2 - 6 = 0.$$

Therefore, this triangle has a right angle at P_2 and it is drawn at Figure 3.
Also, we can use the Pythagoras Theorem to show that $\| Y \|^2 = \| X \|^2 + \| Z \|^2$.

Exercise 3.3.1

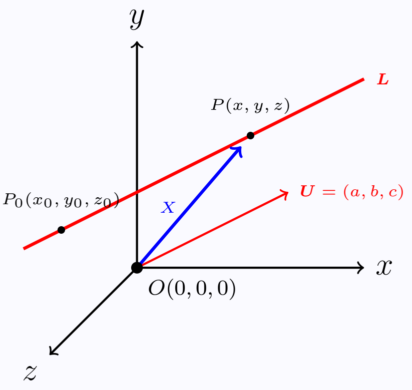
1. Find all values of c so that $X = (c, 2, 1, c)$ and $Y = (c, -1, -2, -3)$ are orthogonal.
2. Show that if X and Y are orthogonal unit vectors in $\mathbb{R}^n$, then
$$\| a\, X + b\, Y \| = \sqrt{a^2 + b^2}.$$
3. Show that if X and Y are orthogonal unit vectors in $\mathbb{R}^n$, then
$$\| 4\, X + 3\, Y \| = 5.$$
4. Let X and Y be two vectors in $\mathbb{R}^n$ so that $\| X \| = \| Y \|$. Show that $X - Y$ and $X + Y$ are orthogonal.
5. Verify that the triangle with vertices $A(1, 1, 2)$, $B(1, 2, 3)$, and $C(3, 0, 3)$ is a right triangle.
6. Find all values of a so that $X = (a^2 - a, -3, -1)$ and $Y = (2, a - 1, 2a)$ are orthogonal.
7. Find a unit vector that is orthogonal to both $X = (1, 1, 0)$ and $Y = (-1, 0, 1)$.
8.

3.4 Lines and Planes in $\mathbb{R}^3$

Definition 3.4.1

A line in $\mathbb{R}^3$ is determined by a fixed point $P_0 = (x_0, y_0, z_0)$ and a directional vector $\boldsymbol{U} = (a, b, c)$. The line $\boldsymbol{L}$ through P_0 and parallel to $\boldsymbol{U}$ consists of the points $P(x, y, z)$ such that

$$(x, y, z) = (x_0, y_0, z_0) + t(a, b, c), \text{ where } t \in \mathbb{R}. \qquad (3.4.1)$$



Such equation is written as $X = P_0 + t\boldsymbol{U}$, where $X = (x, y, z)$. The **parametric equation** of line $\boldsymbol{L}$ (Equation 3.4.1):

$$\left.\begin{array}{rcrcr} x & = & x_0 & + & at \\ y & = & y_0 & + & bt \\ z & = & z_0 & + & ct \end{array}\right\} t \in \mathbb{R},$$

while the **symmetric form** of $\boldsymbol{L}$ is given by:

$$t := \frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}.$$

Example 3.4.1

Let $P_1(2, -2, 3)$, $P_2(-1, 0, 4)$, $P_3(-4, 2, 5)$ be three points in $\mathbb{R}^3$.

1. Find the parametric equation and the symmetric form of the line that passes through the points P_1 and P_2.
2. Does P_3 lies on the same line? Explain.

Solution:

1. Let $\boldsymbol{U} = \overrightarrow{P_1P_2} = P_2 - P_1 = (-3, 2, 1)$ and let $P_0 = P_1$ be a fixed point on the line call it $\boldsymbol{L}$. Then, the parametric equations of $\boldsymbol{L}$ are:

$$\left.\begin{array}{rcrcr} x & = & 2 & - & 3t \\ y & = & -2 & + & 2t \\ z & = & 3 & + & t \end{array}\right\} t \in \mathbb{R}$$

while the symmetric form of $\boldsymbol{L}$ is:

$$\frac{x-2}{-3} = \frac{y-(-2)}{2} = \frac{z-3}{1} \qquad \Longleftrightarrow \qquad \frac{2-x}{3} = \frac{y+2}{2} = z-3.$$

2. We have to check if P_3 satisfies the parametric equation or the symmetric form of $\boldsymbol{L}$:

$$\frac{2-(-4)}{3} = \frac{(2)+2}{2} = (5)-3 =: 2.$$

Therefore, $t=2$ and P_3 lies on $\boldsymbol{L}$. The same check can be done using parametric equations of $\boldsymbol{L}$.

Remark 3.4.1

Let $\boldsymbol{U}_1 = (a_1, b_1, c_1)$ and $\boldsymbol{U}_2 = (a_2, b_2, c_2)$ be two vectors associated with $\boldsymbol{L}_1$ and $\boldsymbol{L}_2$ so that

$$\boldsymbol{L}_1 : \frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1} \text{ and } \boldsymbol{L}_2 : \frac{x-x_2}{a_2} = \frac{y-y_2}{b_2} = \frac{z-z_2}{c_2}. \text{ Then,}$$

1. $\boldsymbol{L}_1 \perp \boldsymbol{L}_2 \Longleftrightarrow \boldsymbol{U}_1 \perp \boldsymbol{U}_2 \Longleftrightarrow \boldsymbol{U}_1 \cdot \boldsymbol{U}_2 = 0,$
2. $\boldsymbol{L}_1 // \boldsymbol{L}_2 \Longleftrightarrow \boldsymbol{U}_1 // \boldsymbol{U}_2 \Longleftrightarrow \boldsymbol{U}_1 \times \boldsymbol{U}_2 = \boldsymbol{0} \Longleftrightarrow \boldsymbol{U}_2 = c\,\boldsymbol{U}_1$ for $c\mathbb{R}$.

Example 3.4.2

Show that $\boldsymbol{L}_1 : P_1(4,-1,4)$ and $\boldsymbol{U}_1 = (1,1,-3)$ and $\boldsymbol{L}_2 : P_2(3,-1,0)$ and $\boldsymbol{U}_2 = (2,1,1)$ intersect orthogonally, and find the intersection point.

Solution:

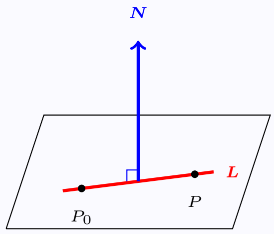
Clearly, $\boldsymbol{U}_1 \cdot \boldsymbol{U}_2 = 2+1-3 = 0$. Then, $\boldsymbol{U}_1 \perp \boldsymbol{U}_2 \Longrightarrow \boldsymbol{L}_1 \perp \boldsymbol{L}_2$. To find the intersection point $P(x,y,z)$, we look for a point satisfying both parametric equations at the same time:

$$\begin{array}{llll} \boldsymbol{L}_1 : & x = 4 + \ \ t_1\ , & y = -1 + t_1, \text{ and } z = 4 - 3t_1, \\ \boldsymbol{L}_2 : & x = 3 + 2t_2\ , & y = -1 + t_2, \text{ and } z = \qquad t_2. \end{array}$$

Clearly, since $y = -1 + t_1 = -1 + t_2$, we get $t_1 = t_2$. Substituting this in $z = 4 - 3t_1 = t_2$, we get $4 - 3t_1 = t_1$ which implies that $t_1 = 1 = t_2$. Therefore, the intersection point according to $\boldsymbol{L}_1$ is $P(4+1, -1+1, 4-3)$, that is $P(5,0,1)$.

Definition 3.4.2

The equation of a **plane $\mathbf{\Pi}$** is determined by a fixed point $P_0(x_0, y_0, z_0)$ contained in $\mathbf{\Pi}$ and a **normal** directional vector $\boldsymbol{N} = (a, b, c)$ which is orthogonal to $\mathbf{\Pi}$. A point $P(x, y, z)$ lies on the plane $\mathbf{\Pi}$ if and only if



$$\boldsymbol{N} \perp \overrightarrow{P_0P} \quad \Longleftrightarrow \quad \boldsymbol{N} \cdot \overrightarrow{P_0P} = 0.$$

The **point-normal equations** (**general form**) of the plane $\mathbf{\Pi}$ that passes through $P_0(x_0, y_0, z_0)$ and its normal vector is $\boldsymbol{N} = (a, b, c)$ is

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$

Where the **standard form** of the plane $\mathbf{\Pi}$ is

$$ax + by + cz + d = 0.$$

Remark 3.4.2

Assume that we want to find an equation of a plane $\mathbf{\Pi}$ containing three points $P_1(x_1, y_1, z_1)$, $P_2(x_2, y_2, z_2)$, and $P_3(x_3, y_3, z_3)$, then we can use either of the following ways:

1. For any point $P(x, y, z) \in \mathbf{\Pi}$, we use the standard form of $\mathbf{\Pi}$: $ax + by + cz + d = 0$ and apply it for the points P_1, P_2, and P_3. This is a homogenous system in a, b, c, and d. This system has non-trivial solutions if

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0.$$

   Solving this determinant, we get an equation in the standard form for $\mathbf{\Pi}$.

2. Another way is to compute two contained vectors in $\mathbf{\Pi}$ namely $X = \overrightarrow{P_1P_2}$ and $Y = \overrightarrow{P_1P_3}$ and consider the normal vector to $\mathbf{\Pi}$ which is $\boldsymbol{N} = X \times Y = (a, b, c)$ which is orthogonal to $\mathbf{\Pi}$. Then, the general form of $\mathbf{\Pi}$ is $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$.

Example 3.4.3

Let $P_1(2,-2,1)$, $P_2(-1,0,3)$, $P_3(5,-3,4)$, and $P_4(4,-3,7)$ be four points in $\mathbb{R}^3$. Then,

1. Find an equation of the plane $\mathbf{\Pi}$ that passes through P_1, P_2, and P_3.
2. Is P_4 contained in $\mathbf{\Pi}$? Explain.

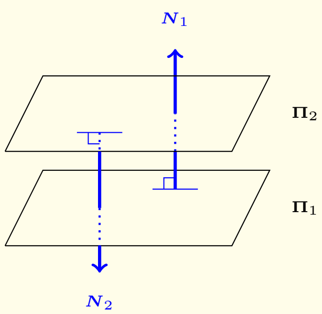
Solution:

1. Let $X = \overrightarrow{P_1P_2} = P_2 - P_1 = (-3,2,2)$ and $Y = \overrightarrow{P_1P_3} = P_3 - P_1 = (3,-1,3)$. These two vectors are contained in $\mathbf{\Pi}$ while $\boldsymbol{N} = X \times Y = (8,15,-3)$ is a normal vector to $\mathbf{\Pi}$. Therefore, a general form of $\mathbf{\Pi}$ is $8(x-2)+15(y+2)-3(z-1)=0$. The standard form of $\mathbf{\Pi}$ is
$$8x+15y-3z+17=0.$$
2. P_4 is contained in $\mathbf{\Pi}$ if it satisfies its equation:
$$8(4)+15(-3)-3(7)+17=-17\neq 0.$$
Therefore, P_4 is **not** on the plane $\mathbf{\Pi}$.

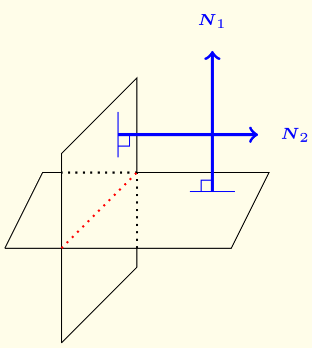
Remark 3.4.3

Let $\mathbf{\Pi}_1 : a_1x+b_1y+c_1z+d_1=0$ and $\mathbf{\Pi}_2 : a_2x+b_2y+c_2z+d_2=0$. Then,

1. $\mathbf{\Pi}_1 // \mathbf{\Pi}_2 \iff \boldsymbol{N}_1 // \boldsymbol{N}_2 \iff \boldsymbol{N}_1 \times \boldsymbol{N}_2 = \mathbf{0} \iff \boldsymbol{N}_2 = c\,\boldsymbol{N}_1,$ where $c \in \mathbb{R}$,
2. $\mathbf{\Pi}_1 \perp \mathbf{\Pi}_2 \iff \boldsymbol{N}_1 \perp \boldsymbol{N}_2 \iff \boldsymbol{N}_1 \cdot \boldsymbol{N}_2 = 0.$



$\mathbf{\Pi}_1 // \mathbf{\Pi}_2$



$\mathbf{\Pi}_1 \perp \mathbf{\Pi}_2$

Example 3.4.4

Find the parametric equation of the intersection line of the two planes:

$$\mathbf{\Pi}_1 : x - y + 2z = 3 \text{ and } \mathbf{\Pi}_2 : 2x + 4y - 2z = -6.$$

Solution:

We form a non-homogenous system to solve for the parametric equation of the intersection line:

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 3 \\ 2 & 4 & -2 & -6 \end{array}\right] \xrightarrow{r_2 - 2r_1 \to r_2} \left[\begin{array}{ccc|c} 1 & -1 & 2 & 3 \\ 0 & 6 & -6 & -12 \end{array}\right] \xrightarrow{\frac{1}{6}r_2 \to r_2} \left[\begin{array}{ccc|c} 1 & -1 & 2 & 3 \\ 0 & 1 & -1 & -2 \end{array}\right] \xrightarrow{r_1 + r_2 \to r_1} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & -2 \end{array}\right]$$

Therefore, the reduced system is:

$$x + z = 1 \text{ and } y - z = -2.$$

Let $z = t \in \mathbb{R}$ to get the parametric equation of the intersection line:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 - t \\ -2 + t \\ t \end{bmatrix}$$

Example 3.4.5

Find two equations of two planes whose intersection line is the line $\boldsymbol{L}$:

$$x = -2 + 3t;\ y = 3 - 2t;\ z = 5 + 4t; \text{ where } t \in \mathbb{R}.$$

Solution:

The symmetric form of $\boldsymbol{L}$ is:

$$\frac{x+2}{3} = \frac{y-3}{-2} = \frac{z-5}{4}.$$

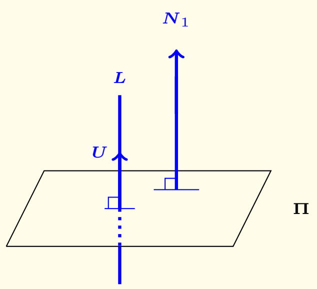
Therefore, a first plane is by equating $\frac{x+2}{3} = \frac{y-3}{-2}$, to get $-2x-4 = 3y-9$. Thus, $\mathbf{\Pi}_1 : 2x+3y-5 = 0$. Another plane is by equating $\frac{x+2}{3} = \frac{z-5}{4}$, to get $4x+8 = 3z-15$. Thus $\mathbf{\Pi}_2 : 4x-3z+23 = 0$.

Remark 3.4.4

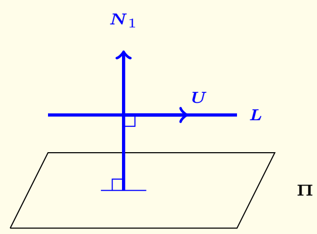
Let $\boldsymbol{U} = (a_1, b_1, c_1)$ be associated with the line $\boldsymbol{L} : \dfrac{x - x_0}{a_1} = \dfrac{y - y_0}{b_1} = \dfrac{z - z_0}{c_1}$ and $\boldsymbol{N} = (a_2, b_2, c_2)$ be associated with the plane $\boldsymbol{\Pi}_2 : a_2x + b_2y + c_2z + d_2 = 0$. Then,

1. $\boldsymbol{L} \perp \boldsymbol{\Pi} \iff \boldsymbol{U} // \boldsymbol{N} \iff \boldsymbol{U} \times \boldsymbol{N} = \mathbf{0} \iff \boldsymbol{U} = c\,\boldsymbol{N}$ where $c \in \mathbb{R}$,

2. $\boldsymbol{L} // \boldsymbol{\Pi} \iff \boldsymbol{U} \perp \boldsymbol{N} \iff \boldsymbol{U} \cdot \boldsymbol{N} = 0$.



$\boldsymbol{L} \perp \boldsymbol{\Pi}$



$\boldsymbol{L} // \boldsymbol{\Pi}$

Example 3.4.6

Find a plane that passes through the point $(2, 4, -3)$ and is parallel to the plane $-2x + 4y - 5z + 6 = 0$.

Solution:

Since the two planes parallel, we can choose the normal vector of the second plane. That is $\boldsymbol{N} = (-2, 4, -5)$. Thus, the equation of the plane is

$$-2(x - 2) + 4(y - 4) - 5(z + 3) = 0 \implies -2x + 4y - 5z - 27 = 0.$$

Example 3.4.7

Find a line that passes through the point $(-2, 5, -3)$ and is perpendicular to the plane $2x - 3y + 4z + 7 = 0$.

Solution:

The line $\boldsymbol{L}$ is perpendicular to our plane. So, it is parallel to its normal vector, so we can choose the normal vector as $\boldsymbol{U}$. That is $\boldsymbol{U} = (2, -3, 4)$ and hence the parametric equation of $\boldsymbol{L}$ is

$$\left.\begin{array}{rcrcr} x & = & -2 & + & 2t \\ y & = & 5 & - & 3t \\ z & = & -3 & + & 4t \end{array}\right\} t \in \mathbb{R}$$

Example 3.4.8

Show that the plane $\boldsymbol{\Pi} : 6x - 4y + 2z = 0$ and the line $\boldsymbol{L} : \frac{x-1}{3} = \frac{-y+4}{2} = z - 5$ intersect orthogonally. Find the intersection point.

Solution:

We first have to write the symmetric form as

$$\boldsymbol{L} : \quad \frac{x-1}{3} = \frac{y-4}{-2} = \frac{z-5}{1}.$$

Then, the normal vector of $\boldsymbol{\Pi}$ is $\boldsymbol{N} = (6, -4, 2)$ and the directional vector of $\boldsymbol{L}$ is $\boldsymbol{U} = (3, -2, 1)$. Clearly, $\boldsymbol{N} = 2\boldsymbol{U}$ which implies that $\boldsymbol{N} // \boldsymbol{U} \iff \boldsymbol{\Pi} \perp \boldsymbol{L}$.

The intersection point with respect to $\boldsymbol{L}$ is

$$\left.\begin{array}{rcrcr} x & = & 1 & + & 3t \\ y & = & 4 & - & 2t \\ z & = & 5 & + & t \end{array}\right\} t \in \mathbb{R}$$

Therefore, plugin these values into the plane equation, we get

$$\begin{array}{rcl} 6(1+3t) - 4(4-2t) + 2(5+t) & = & 0 \\ 6 + 18t - 16 + 8t + 10 + 2t & = & 0 \\ 28t & = & 0 \end{array}$$

Therefore, we get $t = 0$. Substituting this in the parametric equatio, we get the intersection point as $P(1, 4, 5)$.

Exercise 3.4.1

1. Consider the planes:

$$\boldsymbol{\Pi}_1 : \ x+y+z=3, \quad \boldsymbol{\Pi}_2 : \ x+2y-2z=2k, \quad \text{and } \boldsymbol{\Pi}_3 : \ x+k^2z=2.$$

   Find all values of k for which the intersection of the three planes is a line. **Hint:** Any point on the intersection of the three planes must satisfies the three equations. This would give a system of three equation. This system must have infinitely many solutions to describe a line.

2. Consider the lines:

$$\boldsymbol{L}_1 : \ \frac{x+1}{3} = \frac{y+4}{2} = z-1, \quad \text{and} \quad \boldsymbol{L}_2 : \ \frac{x-3}{2} = \frac{y-4}{-4} = \frac{z-2}{2}.$$

   (a) Show that $\boldsymbol{L}_1$ and $\boldsymbol{L}_2$ are perpendicular and find their point of intersection.

   (b) Find an equation of the plane $\boldsymbol{\Pi}$ that contains both $\boldsymbol{L}_1$ and $\boldsymbol{L}_2$.

   Hint: (a) Show that $\boldsymbol{U}_1 \cdot \boldsymbol{U}_2 = 0$ and then find a point satisfying both equations of x, y, and z in terms of t_1 and t_2, for instance. (b) Consider $\boldsymbol{N} = \boldsymbol{U}_1 \times \boldsymbol{U}_2$.

3. Let L be the line through the points $P_1(-4,2,-6)$ and $P_2(1,4,3)$.

   (a) Find parametric equations for L.

   (b) Find two planes whose intersection is L.

4. Find the parametric equations for the line L which passes through the points $P(2,-1,4)$ and $Q(4,4,-2)$. For what value of k is the point $R(k+2,14,-14)$ on the line L?

5. Find the point of intersection of the line $x=1-t, y=1+t, z=t$, and the plane $3x+y+3z-1=0$.

6. Find the equations in symmetric form of line of intersection of planes:

$$\Pi_1 : x+2y-z=2, \quad \text{and} \quad \Pi_2 : 3x+7y+z=11.$$

7. Find an equation of the plane containing the lines

$$L_1 : x=3+t, y=1-t, z=3t, \quad \text{and} \quad L_2 : x=2s, y=-2+s, z=5-s.$$

8. Find $a,b \in \mathbb{R}$ so that the point $P(3, a-2b, 2a+b)$ lies on the line

$$L : x=1+2t, y=2-t, z=4+3t.$$

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