Lecture Notes in Euclidean Geometry: Math 226

Dr. Abdullah Al-Azemi

Mathematics Department
Kuwait University

September 6, 2019

Contents

Chapter

1 Triangles and Quadrilaterals

In this chapter, we discuss the following topics in some details: Lines and angles; Parallelism; Congruencey and similarity of triangles; Isosceles and equilateral triangles; Right-angled triangles; Parallelogram; Rhombus; Rectangle; and Square.

1.1 Lines and Angles

Any two **point**s A and B determine a unique **line** l, denoted by $\overleftrightarrow{AB}$. Two lines l and m intersect in at most one point. If l do not intersect m, then we say that l and m are parallel lines, denoted $\overleftrightarrow{l} \parallel \overleftrightarrow{m}$. On the other hand, if two (or more) lines intersect in one point, the lines are said to be **concurrent**. Moreover, points on one line are called **collinear**.

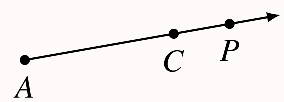
Theorem 1.1.1

If l is a line and P is a point not on l, then:

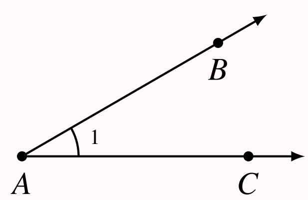
1. There is exactly one line through P that is parallel to l.
2. There is exactly one line through P that is perpendicular to l.

A **bisector** of a **segment** is a line intersecting the segment at its midpoint. A **perpendicular bisector** of a segment is a line that is perpendicular to the segment at its midpoint. As a reslt, any point lies on the perpendiclar bisector is **equidistant** (has equal distant) from the endpoints of the segment.

A **ray** AC, denoted $\overrightarrow{AC}$, consists of the segment $\overline{AC}$ and all other points P such that C is between A and P. In this case, A is called the **endpoint** of the ray.



An **angle** $\hat{A}$ is formed by two rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$ that have the same end point A. The rays then are called the **sides** of the angle, and A is called the **vertex** of the angle. In the diagram, the angle can be denoted: $\hat{A}$, $B\hat{A}C$, $C\hat{A}B$, or $\hat{1}$.

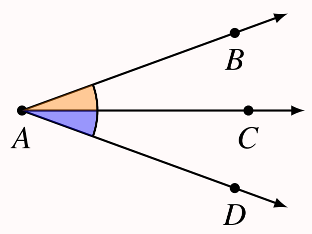


There are several types of angles:

1. **Acute** angle: measures between 0° and 90°.
2. **Right** angle: measures exactly 90°.
3. **Obtuse** angle: measures between 90° and 180°.
4. **Straight** angle: measures exactly 180°.

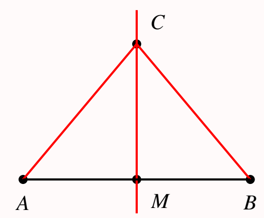
Two angles $\hat{A}$ and $\hat{B}$ with equal measures are called **congruent** angles, denoted $\hat{A} \cong \hat{B}$.

Two angles with a common vertex and a common side are called **adjacent** angles. The **bisector** of an angle is the ray that divides the angle into two congruent adjacent angles. As a reslt, a point lies on the bisector of an angle **if and only if** it is equidistant (has equal distant) from the sides of the angle. In the diagram: The distance between C and B equals the distance between C and D.



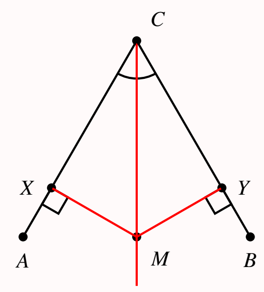
Theorem 1.1.2

A point lies on the perpendicular bisector of a segment if and only if the point is equidistant from the end point of the segment.

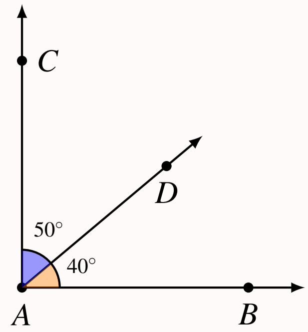


Theorem 1.1.3

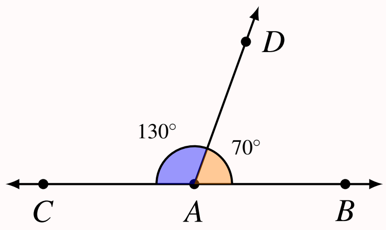
A point lies on the angle bisector of an angle if and only if the point is equidistant from the sides of the angle.



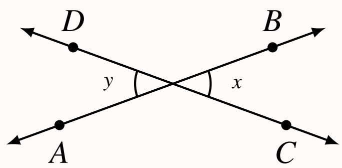
Complementary angles are two angles whose measures have the sum 90°. Each angle is called **complement** of each other.



Supplementary angles are two angles whose measures have the sum 180°. Each angle is called **supplement** of each other.



Vertical angles (vert.) are two angles such that the sides of one angle are opposite rays to the sides of the other angle.



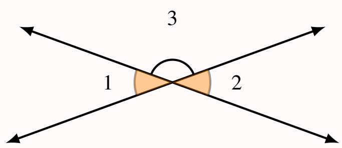
Theorem 1.1.4

Vertical angles are congruent.

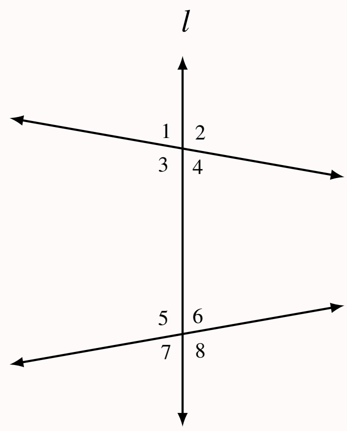
Proof:

Note that angles $\hat{1}$ and $\hat{3}$; and angles $\hat{2}$ and $\hat{3}$ are both supplementary angles. That is

$$180^\circ = |\hat{1}| + |\hat{3}| = |\hat{2}| + |\hat{3}|. \text{ Therefore, } |\hat{1}| = |\hat{2}|.$$



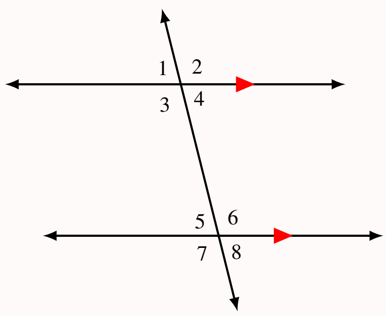
- A **transversal** is a line cutting off two or more other lines in different points. Example: l is a transversal.
- **Alternate interior** angles are two nonadjacent interior angles on opposite sides of a transversal. Example: $\hat{3}$ and $\hat{6}$; $\hat{4}$ and $\hat{5}$ are alternate interior angles.
- **Same-side interior** angles are two interior angles on the same side side of the transversal. Example: $\hat{3}$ and $\hat{5}$; $\hat{4}$ and $\hat{6}$ are same-side interior angles.
- **Corresponding** angles are two angles in corresponding position relative to the two intersected lines. Example: $\hat{1}$ and $\hat{5}$; $\hat{2}$ and $\hat{6}$; $\hat{3}$ and $\hat{7}$; $\hat{4}$ and $\hat{8}$ are corresponding angles.



Theorem 1.1.5

If two lines are cut off by a transversal, then the two lines are parallel if and only if any of the following hold:

1. Corresponding angles are congruent. e.g. $\hat{1} \cong \hat{5}$,
2. Altrnate interior angles are congruent. e.g. $\hat{3} \cong \hat{6}$, or
3. Same-side interior angles are supplementary. e.g. $|\hat{3}| + |\hat{5}| = 180°$.

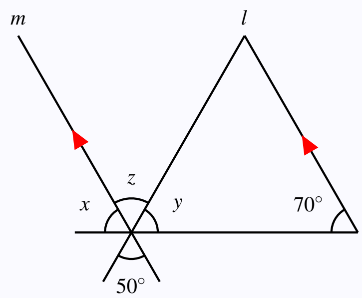


Example 1.1.1

Find the values of x and y in the diagram.

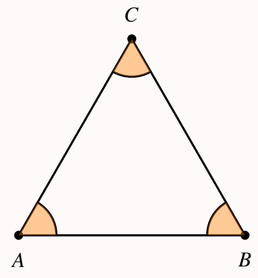
Solution:

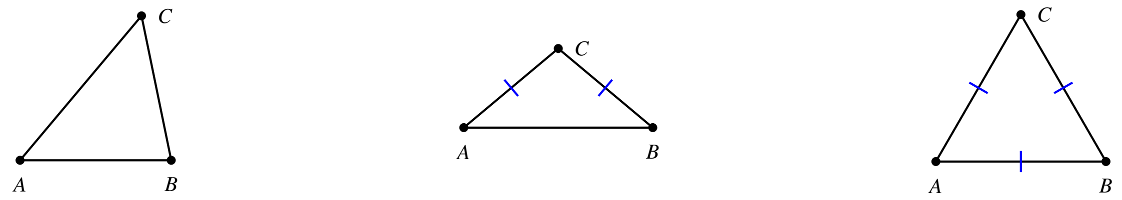
Observe that we have two parallel lines: $\overleftrightarrow{m} \parallel \overleftrightarrow{l}$. Then, by Theorem 1.1.5, we have $|\hat{x}| = 70°$ (since they are corresponding angles). Also, Theorem 1.1.4 implies that $|\hat{z}| = 50°$ (they are opposite angles). Note that, angles $\hat{x}$, $\hat{A}$, and $\hat{y}$ are supplementary, and hence $|\hat{y}| = 180° - |\hat{x}| - |\hat{z}| = 60°$.



A **triangle** is formed by three segments called **sides**, and three points each is called a **vertex**.

- Triangle ABC is denoted $\triangle\, ABC$.
- Vertices of $\triangle\, ABC$: A, B, and C.
- Sides of $\triangle\, ABC$: $\overline{AB}$, $\overline{AC}$, and $\overline{BC}$.
- Angles of $\triangle\, ABC$: $\hat{A}$, $\hat{B}$, and $\hat{C}$.



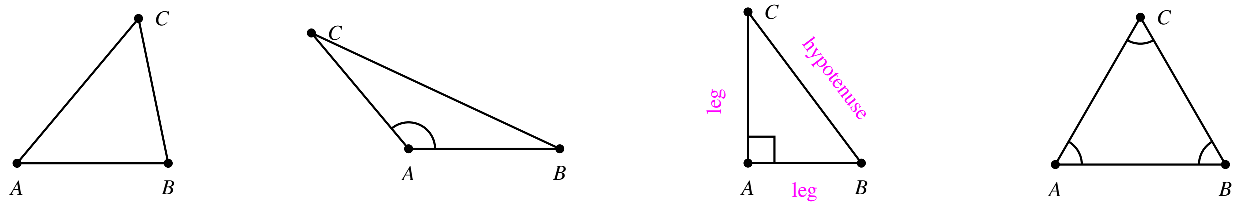


Scalene. No congruent sides.

Isosceles. At least two sides congruent.

Equilateral. All sides congruent.

Figure 1.1: Types of triangles with respect to their sides congruence.



Acute. Three acute angles.

Obtuse. One obtuse angle.

Right. One right angle.

Equiangular. All angles congruent.

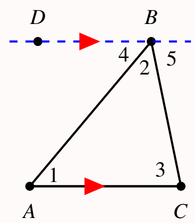
Figure 1.2: Types of triangles with respect to their angles.

Theorem 1.1.6

The measure of angles of any triangle sums to $180°$.

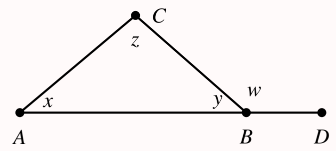
Proof:

Let $\triangle\, ABC$ be any triangle. Draw a line $\overleftrightarrow{BD}$ parallel to $\overleftrightarrow{AC}$, see the diagram. Note that $|\hat{2}| + |\hat{4}| + |\hat{5}| = 180°$ (supp. angles). The line $\overleftrightarrow{AB}$ is a transversal to the parallel lines $\overleftrightarrow{BD}$ and $\overleftrightarrow{AC}$. Hence, $\hat{1} \cong \hat{4}$ (alternate interior angles). Also, $\overleftrightarrow{BC}$ is another transversal and hence $\hat{3} \cong \hat{5}$. Thus, $|\hat{2}| + |\hat{1}| + |\hat{3}| = |\hat{2}| + |\hat{4}| + |\hat{5}| = 180°$.

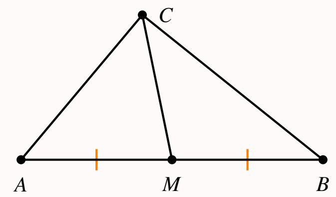


When one side of a triangle is extended, an **exterior** angle is formed.

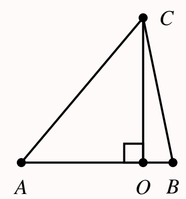
Fact: The measure of the exterior angle equal the sum of the other two nonadjacent angles of the triangle. That is, $|\hat{w}| = |\hat{x}| + |\hat{z}|$.



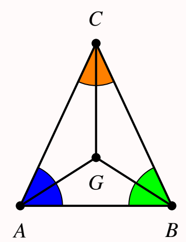
A **median** $\overline{CM}$ of a triangle is a segment from a vertex to the middle point of opposite side (inside the triangle).



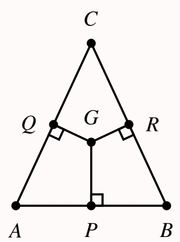
An **altitude** $\overline{CO}$ of a triangle is the perpendicular segment from a vertex to the line that contains the opposite side (might be inside or outside the triangle).



An **incenter** is the point of intersection of the angle bisectors of a triangle.



A **circumcenter** is the point of intersection of the perpendicular bisectors of the sides of a triangle.



1.2 Congruent and Similar Triangles

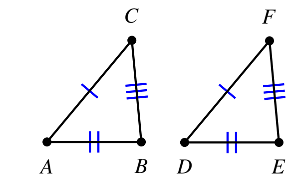
Definition 1.2.1

Two triangles are **congruent** if and only if their vertices can be matched up so that the corresponding parts (sides and angles) of the triangles are congruent. In that case, we write the corresponding vertices in the same order. That is, $\triangle ABC \cong \triangle DEF$ means that $\hat{A} \cong \hat{D}$, $\hat{B} \cong \hat{E}$, and $\hat{C} \cong \hat{F}$; and $\overline{AB} \cong \overline{DE}$, $\overline{AC} \cong \overline{DF}$, and $\overline{BC} \cong \overline{EF}$.

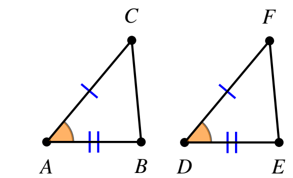
Remark 1.2.1: Showing Two Triangles are Congruent

Two triangles are congruent if any condition of the following holds:

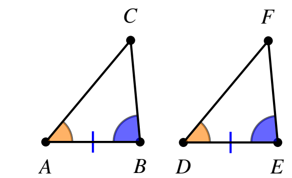
1. **S.S.S.**: The three sides of two triangles are congruent.
2. **S.A.S.**: Two sides and the included angle of two triangles are congruent.
3. **A.S.A.**: Two angles and the included side of two triangles are congruent.
4. **A.A.S.**: Two angles and a non-included side of two triangles are congruent.
5. **H.L.**: The hypotenuse and a leg of two **(right)** triangles are congruent.



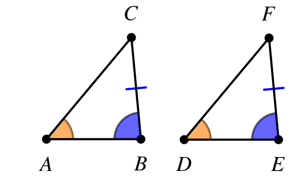
1. S. S. S.
$\triangle ABC \cong \triangle DEF$ by SSS.



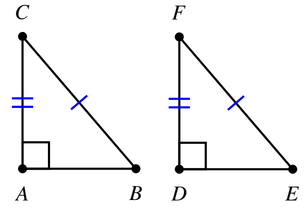
2. S. A. S.
$\triangle ABC \cong \triangle DEF$ by SAS.



3. A. S. A.
$\triangle ABC \cong \triangle DEF$ by ASA.



4. A. A. S.
$\triangle ABC \cong \triangle DEF$ by AAS.



5. H. L.
$\triangle ABC \cong \triangle DEF$ by HL.

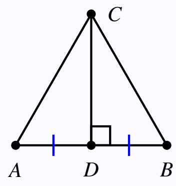
Example 1.2.1

Let D be the midpoint of $\overline{AB}$, and let $\overline{CD} \perp \overline{AB}$. Prove: $\triangle ADC \cong \triangle BDC$.

Solution:

1. $\overline{AD} \cong \overline{BD}$ (because D is a midpoint of $\overline{AB}$).
2. $A\hat{D}C \cong B\hat{D}C$, because $\overline{CD} \perp \overline{AB}$.
3. $\overline{CD}$ is common to triangles $\triangle ADC$ and $\triangle BDC$.

Therefore, $\triangle ADC \cong \triangle BDC$ by SAS.



Example 1.2.2

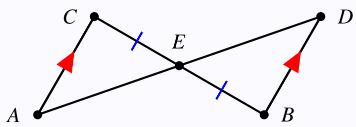
Given: $\overline{CE} \cong \overline{BE}$ and $\overline{AC} \parallel \overline{BD}$. Prove: $\triangle ACE \cong \triangle DBE$.

Solution:

Note that $\overleftrightarrow{BC}$ is a transversal to the parallel lines $\overleftrightarrow{AC}$ and $\overleftrightarrow{BD}$

1. $\hat{B} \cong \hat{C}$ (alt. int. angles).
2. $\overline{CE} \cong \overline{BE}$ (Given).
3. $B\hat{E}D \cong A\hat{E}C$ (vertical opposite angles).

Therefore, by ASA, $\triangle ACE \cong \triangle DBE$.



A **proportion** is an equation $\frac{a}{b} = \frac{c}{d} = k$ (k is called the **scale factor**) stating that the two ratios are equal.

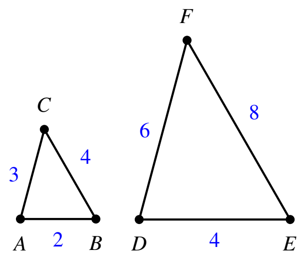
Definition 1.2.2

Two triangle are **similar** if and only if their vertices can be paired so that:

1. corresponding angles are congruent, and
2. corresponding sides are in proportion. (Their lengths have the same ratio).

That is, if $\triangle ABC$ is similar to triangle $\triangle DEF$, we write $\triangle ABC \sim \triangle DEF$ which implies that $\hat{A} \cong \hat{D}$, $\hat{B} \cong \hat{E}$, and $\hat{C} \cong \hat{F}$; and $\frac{|\overline{AB}|}{|\overline{DE}|} = \frac{|\overline{AC}|}{|\overline{DF}|} = \frac{|\overline{BC}|}{|\overline{EF}|}$.

For instance the triangles $\triangle\, ABC$ and $\triangle\, DEF$ are similar:



Remark 1.2.2: Showing Two Triangles are Similar

Two triangles are similar if any condition of the following holds:

1. **S-S.S.S.**: The three sides of two triangles are in proportion.
2. **S-S.A.S.**: Two sides (in proportion) and the included angle (congruent) of two triangles.
3. **S-A.A.**: Two angles (and hence the third) of two triangles are congruent.

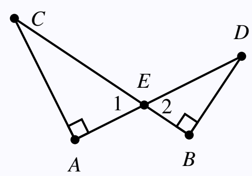
Example 1.2.3

Given: $\hat{A} \cong \hat{B}$ (right angles). Prove: $\triangle\, ACE \sim \triangle\, BDE$. Or Show that $|\overline{AC}| \cdot |\overline{DE}| = |\overline{BD}| \cdot |\overline{CE}|$.

Solution:

1. $\hat{1} \cong \hat{2}$ (vertical opposite angles).
2. Moreover, $\hat{A} \cong \hat{B}$ (given).

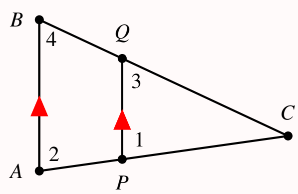
Therefore, $\triangle\, ACE \sim \triangle\, BDE$ by S-AA. That is, $\dfrac{|\overline{AC}|}{|\overline{BD}|} = \dfrac{|\overline{CE}|}{|\overline{DE}|}$ and hence the results.



Theorem 1.2.1: Triangle Proportionality Theorem

If a line parallel to one side of a triangle intersects the other two sides, then it divides those sides proportionally. In fact it produces two similar triangles. That is,

Given: $\triangle\, ABC$; $\overline{PQ} \parallel \overline{AB}$. Then: $\dfrac{|\overline{AP}|}{|\overline{PC}|} = \dfrac{|\overline{BQ}|}{|\overline{QC}|}$. That is $\triangle\, ABC \sim \triangle\, PQC$.



Proof:

Since $\overline{PQ} \parallel \overline{AB}$, we have (corresponding angles) $\hat{1} \cong \hat{2}$ and $\hat{3} \cong \hat{4}$. Since $\hat{C}$ is a common angle in triangles $\triangle ABC$ and $\triangle PQC$, we get $\triangle ABC \sim \triangle PQC$, by S-AA. That is, $\dfrac{|\overline{AC}|}{|\overline{PC}|} = \dfrac{|\overline{BC}|}{|\overline{QC}|}$ where $|\overline{AC}| = |\overline{AP}| + |\overline{PC}|$, and $|\overline{BC}| = |\overline{BQ}| + |\overline{QC}|$. Thus,

$$\frac{|\overline{AP}| + |\overline{PC}|}{|\overline{PC}|} = \frac{|\overline{BQ}| + |\overline{QC}|}{|\overline{QC}|} \quad \Rightarrow \quad \frac{|\overline{AP}|}{|\overline{PC}|} = \frac{|\overline{BQ}|}{|\overline{QC}|}.$$

Theorem 1.2.2: Triangle Angle-Bisector Theorem

The bisector of an angle in a triangle divides the opposite side into segments proportional to the other sides. That is,

Given: $\triangle ABC$; bisector $\overline{CM}$. Then: $\dfrac{|\overline{AC}|}{|\overline{BC}|} = \dfrac{|\overline{AM}|}{|\overline{BM}|}$.

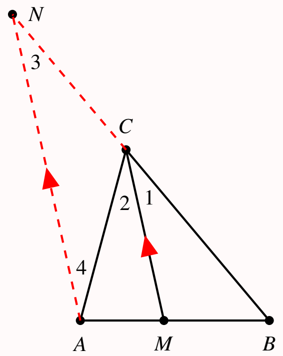
Proof:

Draw $\overline{AN} \parallel \overline{MC}$ so that $\overline{BC}$ intersects $\overline{AN}$ in point N. Then

1. $\hat{1} \cong \hat{2}$ ($\overline{CM}$ is bisector of $\hat{C}$).
2. $\hat{2} \cong \hat{4}$ (alternate interior angles since $\overline{AC}$ is a transversal).
3. $\hat{1} \cong \hat{3}$ (corresponding angles since $\overline{BN}$ is a transversal).

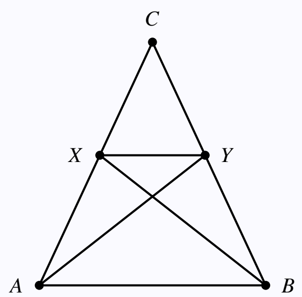
Therefore, $\hat{3} \cong \hat{4}$ and hence $\triangle CNA$ is isosceles with $\overline{NC} \cong \overline{AC}$.

By Theorem 1.2.1, $\triangle BCM \sim \triangle BNA$, and $\dfrac{|\overline{NC}|}{|\overline{BC}|} = \dfrac{|\overline{AM}|}{|\overline{BM}|}$. But $|\overline{NC}| = |\overline{AC}|$. Therefore, $\dfrac{|\overline{AC}|}{|\overline{BC}|} = \dfrac{|\overline{AM}|}{|\overline{BM}|}$.



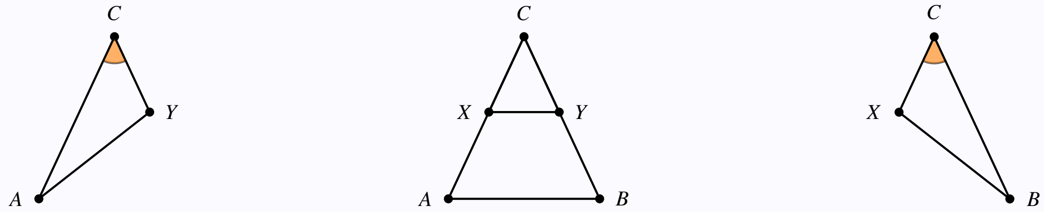
Example 1.2.4

Given: $\triangle AYC \sim \triangle BXC$. Then: $\triangle CXY \sim \triangle CBA$.



Solution:

We first redraw the needed triangles:



It is clear that the angle $\hat{C}$ is a common angle in the two triangles $\triangle\, CXY$ and $\triangle\, CBA$. Since, $\triangle\, AYC \sim \triangle\, BXC$, we have $\dfrac{|\overline{BX}|}{|\overline{AY}|} = \dfrac{|\overline{CX}|}{|\overline{CY}|} = \dfrac{|\overline{CB}|}{|\overline{CA}|}$. That is $\dfrac{|\overline{CX}|}{|\overline{CB}|} = \dfrac{|\overline{CY}|}{|\overline{CA}|}$. Therefore, by S-SAS, we have $\triangle\, CXY \sim \triangle\, CBA$.

1.3 More on Triangles

Theorem 1.3.1

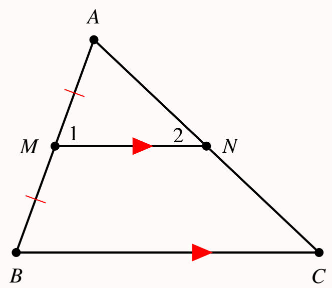
In any triangle $\triangle ABC$, let M be a midpoint of $\overline{AB}$. Then, $\overline{BC} \parallel \overline{MN}$ if and only if N is the midpoint of $\overline{AC}$.

Given: $\triangle ABC$; M is midpoint of $\overline{AB}$. Then: $\overline{BC} \parallel \overline{MN}$ iff N is midpoint of $\overline{AC}$.

Proof:

" $\Rightarrow$": Suppose $\overline{BC} \parallel \overline{MN}$. We show that $\triangle ABC \sim \triangle AMN$.

1. $\hat{1} \cong \hat{B}$ and $\hat{2} \cong \hat{C}$ (corresponding angles).
2. $\hat{A}$ is common.



Thus, by S-AA, we have $\triangle ABC \sim \triangle AMN$.

Hence, $\dfrac{|\overline{AN}|}{|\overline{AC}|} = \dfrac{|\overline{AM}|}{|\overline{AB}|} = \frac{1}{2}$. Therefore, $|\overline{AN}| = \frac{1}{2}|\overline{AC}|$ and therefore, N is the midpoint of $\overline{AC}$.

" $\Leftarrow$": Suppose N is the midpoint of $\overline{AC}$.

Clearly, $\triangle ABC \sim \triangle AMN$ since

1. $\hat{A}$ is common angle.
2. $\dfrac{|\overline{AM}|}{|\overline{AB}|} = \dfrac{|\overline{AN}|}{|\overline{AC}|} = \frac{1}{2}$.

Thus, by S-SAS, we have $\triangle ABC \sim \triangle AMN$.

Therefore, $\hat{1} \cong \hat{B}$ (corresponding angles) which implies that $\overline{MN} \parallel \overline{BC}$.

Theorem 1.3.2

In any triangle $\triangle ABC$, the three angle bisectors concurrent at an equidistant point (called **incenter**) from the sides of the triangle.

Given: $\triangle ABC$; the bisectors of $\hat{A}$, $\hat{B}$, and $\hat{C}$. Then: The angle bisectors intersect in a point; that point is equidistant from $\overline{AB}$, $\overline{AC}$, and $\overline{BC}$.

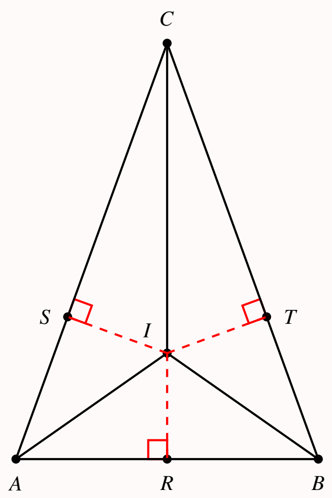
Proof:

Let I be the intersection of bisectors of angles $\hat{A}$, and $\hat{B}$. We show that I also lies on bisector of angle $\hat{C}$; and that I is equidistant from all sides.

Draw segments $\overline{IR}$, $\overline{IS}$, and $\overline{IT}$ perpendicular to $\overline{AB}$, $\overline{AC}$, $\overline{BC}$, respectively. By Theorem 1.1.3, we have

1. $\overline{IR} \cong \overline{IS}$ (I lies on bisector of angle $\hat{A}$).
2. $\overline{IR} \cong \overline{IT}$ (I lies on bisector of angle $\hat{B}$).

Hence $\overline{IS} \cong \overline{IT}$. Again by Theorem 1.1.3, we have I lies on the bisector of angle $\hat{C}$. Clearly, $\left|\overline{IR}\right| = \left|\overline{IS}\right| = \left|\overline{IT}\right|$ and hence I is equidistant from the sides of $\triangle ABC$.



Theorem 1.3.3

In any triangle $\triangle ABC$, the three perpendicular bisectors of the sides concurrent at an equidistant point (called **circumcenter**) from the vertices of the triangle.

Given: $\triangle ABC$; the perpendicular bisectors of $\overline{AB}$, $\overline{AC}$, and $\overline{BC}$. Then: The perpendicular bisectors intersect in a point; that point is equidistant from vertices A, B, and C.

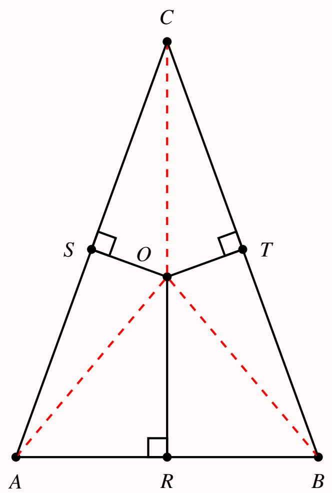
Proof:

Let O be the intersection of perpendicular bisectors of $\overline{AB}$, and $\overline{AC}$. We show that O also lies on perp. bisector of $\overline{BC}$; and that O is equidistant from all vertices.

Draw segments $\overline{OR}$, $\overline{OS}$, and $\overline{OT}$. By Theorem 1.1.2, we have

1. $\overline{OA} \cong \overline{OB}$ (O lies on perp. bisector of $\overline{AB}$).
2. $\overline{OA} \cong \overline{OC}$ (O lies on perp. bisector of $\overline{AC}$).

Hence $\overline{OB} \cong \overline{OC}$. Again by Theorem 1.1.2, we have O lies on the perp. bisector of $\overline{BC}$. Clearly, $\left|\overline{OA}\right| = \left|\overline{OB}\right| = \left|\overline{OC}\right|$ and hence O is equidistant from the vertices of $\triangle ABC$.

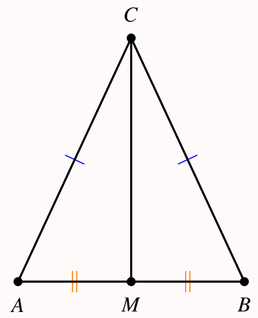


Theorem 1.3.4: The Isosceles Triangle Theorem

The base angles of an isosceles triangle are congruent.

Proof:

Given $\triangle ABC$ isosceles with $\left|\overline{AC}\right| = \left|\overline{BC}\right|$. Let M be the midpoint of $\overline{AB}$. Hence $\overline{AM} \cong \overline{BM}$. Also note that $\overline{CM}$ is common in the two triangles $\triangle ACM$ and $\triangle BCM$. Therefore, $\triangle ACM \cong \triangle BCM$ by SSS. Hence $\hat{A} \cong \hat{B}$.



Theorem 1.3.5: The Converse of Isosceles Triangle Theorem

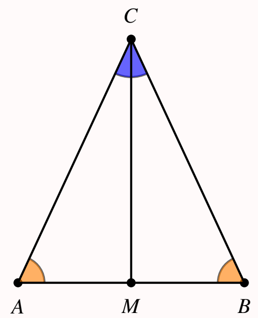
If two angles of a triangle are congruent, then the sides opposite those angles are congruent.

Proof:

Given $\triangle ABC$ with $\hat{A} \cong \hat{B}$. Draw the angle bisector $\overline{CM}$ to get $A\hat{C}M \cong B\hat{C}M$. By AAS, we have $\triangle ACM \cong \triangle BCM$ since

1. $\hat{A} \cong \hat{B}$ (given).
2. $A\hat{C}M \cong B\hat{C}M$ (constructed).
3. $\overline{CM}$ is common.

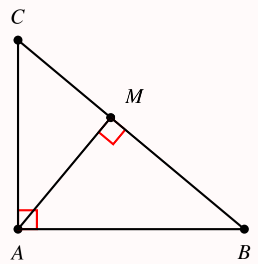
Therefore, $\overline{AC} \cong \overline{BC}$.



Theorem 1.3.6: The Altitude Theorem

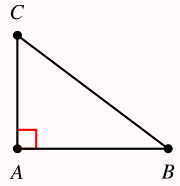
If the altitude is drawn to the hypotenuse of a right triangle, then the two triangles formed are similar to the original one and to each other.

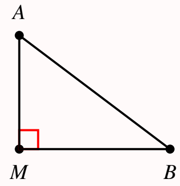
Given: $\triangle ABC$ with right angle $\hat{A}$; altitude $\overline{AM}$. Then: $\triangle BAC \sim \triangle BMA \sim \triangle AMC$.

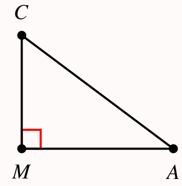


Proof:

Simply redraw the three triangles and use the S-AA to show the similarity.







Theorem 1.3.7: The Pythagorean Theorem

In a right triangle, the square of the hypotenuse equals the sum of the squares of the legs.

Given: Right $\triangle ABC$; $|\overline{A}| = 90°$. Then: $a^2 = b^2 + c^2$.

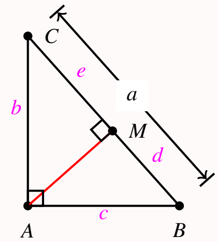
Proof:

By Theorem 1.3.6, we have $\triangle BAC \sim \triangle BMA \sim \triangle AMC$. Hence, we have

$$\frac{|\overline{BA}|}{|\overline{BM}|} = \frac{|\overline{BC}|}{|\overline{BA}|} \quad \Rightarrow \quad \frac{c}{d} = \frac{a}{c} \quad \Rightarrow \quad c^2 = ad.$$

Also, $\dfrac{|\overline{BC}|}{|\overline{AC}|} = \dfrac{|\overline{AC}|}{|\overline{MC}|}$ that is $\dfrac{a}{b} = \dfrac{b}{e}$ and hence $b^2 = ae$. Therefore,

$$b^2 + c^2 = ae + ad = a(d+e) = a^2.$$



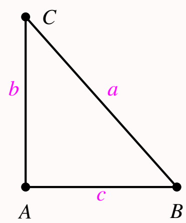
Theorem 1.3.8: The Converse of Pythagorean Theorem

If the square of the one side of a triangle equals the sum of the squares of the two other sides, then the triangle is right.

Given: triangle $\triangle ABC$; $a^2 = b^2 + c^2$. Then: $\triangle ABC$ is right triangle.

Proof:

Let $\triangle DEF$ be a right triangle with legs b and c and the length of hypotenuse is d. Then $d^2 = b^2 + c^2 = a^2$. That is $a = d$. By SSS, $\triangle ABC \cong \triangle DEF$. That is $\triangle ABC$ is a right triangle.



1.4 Parallelograms

Definition 1.4.1

A **parallelogram** ($\square$) is **quadrilateral** (a **polygon** with four sides) with both pairs of opposite sides parallel.

Theorem 1.4.1

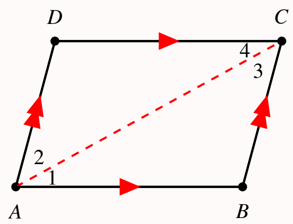
Opposite sides of a parallelogram are congruent. Given: $\square ABCD$. Then: $\overline{AB} \cong \overline{CD}$ and $\overline{AD} \cong \overline{BC}$.

Proof:

Consider $\triangle ABC$ and $\triangle CDA$:

1. $\overline{AC}$ is common.
2. $\hat{1} \cong \hat{4}$ (alternate interior angles).
3. $\hat{2} \cong \hat{3}$ (alternate interior angles).

By ASA: $\triangle ABC \cong \triangle CDA$. Hence $\overline{AB} \cong \overline{CD}$ and $\overline{AD} \cong \overline{BC}$.



Theorem 1.4.2

Opposite angles of a parallelogram are congruent.

Theorem 1.4.3

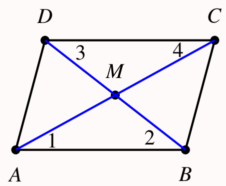
The diagonals of a parallelogram bisect each other.

Given: $\square ABCD$ with diagonals $\overline{AC}$ and $\overline{BD}$. Then: $\overline{AC}$ and $\overline{BD}$ bisect each other.

Proof:

Consider $\triangle AMB$ and $\triangle CMD$. By Theorem 1.4.1, we have $\overline{AB} \cong \overline{CD}$. Also, $\hat{1} \cong \hat{4}$ and $\hat{2} \cong \hat{3}$ (alternate interior angles).

By ASA: $\triangle AMB \cong \triangle CMD$. Hence, $\overline{AM} \cong \overline{CM}$ and $\overline{BM} \cong \overline{DM}$.



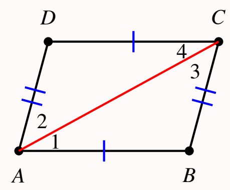
Theorem 1.4.4

In a quadrilateral, if the opposite sides congruent, then it is parallelogram.

Given: $\overline{AB} \cong \overline{CD}$ and $\overline{AD} \cong \overline{BC}$. Then: $\square ABCD$ is parallelogram.

Proof:

By SSS, $\triangle ABC \cong \triangle CDA$. Hence $\hat{1} \cong \hat{4}$ (and $\hat{2} \cong \hat{3}$). By Theorem 1.1.5, we have $\overline{AB} \parallel \overline{CD}$ and $\overline{AD} \parallel \overline{BC}$. Thus, $\square ABCD$ is a parallelogram.



Theorem 1.4.5

In a quadrilateral, if two opposite sides are congruent and parallel, then it is parallelogram.

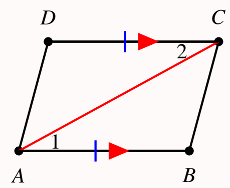
Given: $\overline{AB} \parallel \overline{CD}$ and $\overline{AB} \cong \overline{CD}$. Then: $\square ABCD$ is parallelogram.

Proof:

Consider $\triangle ABC$ and $\triangle CDA$:

1. $\overline{AC}$ is common.
2. $\overline{AB} \cong \overline{CD}$ (given).
3. $\hat{1} \cong \hat{2}$ (alternate interior angles).

By SAS: $\triangle ABC \cong \triangle CDA$. Hence $\overline{AD} \cong \overline{BC}$. Thus, $\square ABCD$ is a parallelogram.



Theorem 1.4.6

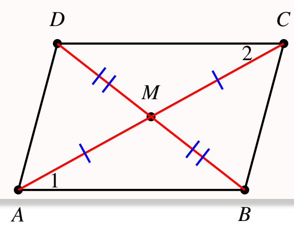
If the diagonals bisect each other in a quadrilateral, then it is parallelogram.

Given: diagonals bisect each other in quadrilateral $ABCD$. Then: $\square ABCD$ is parallelogram.

Proof:

Consider $\triangle ABM$ and $\triangle CDM$:

1. $A\hat{M}B \cong C\hat{M}D$ (vertically opposite).
2. $\overline{AM} \cong \overline{CM}$ (given).
3. $\overline{BM} \cong \overline{DM}$ (given).



By SAS: $\triangle ABM \cong \triangle CDM$. Then $\overline{AB} \cong \overline{CD}$ and $\hat{1} \cong \hat{2}$ which implies that $\overline{AB} \parallel \overline{CD}$. That is $ABCD$ is a parallelogram.

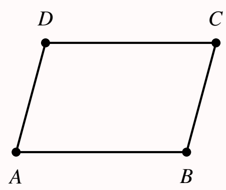
Theorem 1.4.7

If the opposite angles are congruent in a quadrilateral, then it is parallelogram.

Given: $\hat{A} \cong \hat{C}$ and $\hat{B} \cong \hat{D}$ in quadrilateral $ABCD$. Then: $\square ABCD$ is parallelogram.

Proof:

$|\hat{A}|+|\hat{B}|+|\hat{C}|+|\hat{D}| = 2|\hat{A}|+2|\hat{B}| = 360$. That is, $|\hat{A}|+|\hat{B}| = 180$ ($\hat{A}$ and $\hat{B}$ are supplementary). By Theorem 1.1.5, $\overline{AD} \parallel \overline{BC}$. But then $\hat{A}$ and $\hat{D}$ are also supplementary and again $\overline{AB} \parallel \overline{CD}$. That is $ABCD$ is a parallelogram.

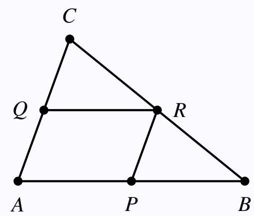


Example 1.4.1

Let $\triangle ABC$ be a triangle with P, Q, and R are midpoints for $\overline{AB}$, $\overline{AC}$, and $\overline{BC}$, respectively. Show that $APRQ$ is a parallelogram.

Solution:

By Theorem 1.3.1, we have $\overline{AP} \parallel \overline{QR}$ and $\overline{AC} \parallel \overline{PR}$. Hence, $\overline{AP} \parallel \overline{QR}$ and $\overline{AQ} \parallel \overline{PR}$. That is $APRQ$ is a parallelogram.



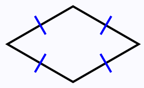
1.5 Special Parallelograms

Definition 1.5.1

A **rectangle** is a parallelogram with four right angles.

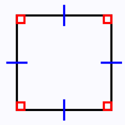
Definition 1.5.2

A **rhombus** is a parallelogram with four congruent sides.



Definition 1.5.3

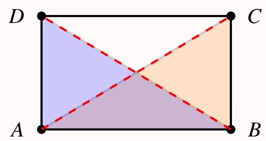
A **square** is a parallelogram with four congruent sides and four right angles. Thus, every square is a rectangle and a rhombus.

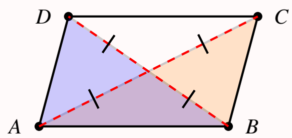


Theorem 1.5.1

Let $ABCD$ be a parallelogram, then $ABCD$ is a rectangle if and only if its diagonals are congruent.

Proof:





„ $\Rightarrow$": Suppose that $ABCD$ is a rectangle. Then it has four right angles. In the two triangles $\triangle\, ABC$ and $\triangle\, BAD$, we have:

1. $\overline{AB}$ is common.
2. $A\hat{B}C \cong B\hat{A}D$ (both are right).
3. $\overline{AD} \cong \overline{BC}$ (It is parallelogram).

By SAS: $\triangle\, ABC \cong \triangle\, CDA$. Hence $\overline{AC} \cong \overline{BD}$.

„ $\Leftarrow$": Suppose that $ABCD$ is a parallelogram with congruent diagonal $\overline{AC}$ and $\overline{BD}$. In the two triangles $\triangle\, ABC$ and $\triangle\, BAD$, we have:

1. $\overline{AB}$ is common.

2. $\overline{AD} \cong \overline{BC}$ (It is a parallelogram).
3. $\overline{AC} \cong \overline{BD}$ (given).

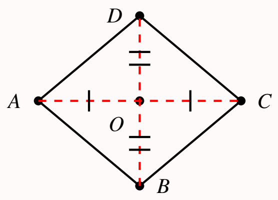
By SSS: $\triangle ABC \cong \triangle BAD$. Thus $\hat{A} \cong \hat{B}$. But since $\overline{AD} \parallel \overline{BC}$, we have $|\hat{A}| + |\hat{B}| = 180°$ (same-side interior angles are supplementary). Hence $|\hat{A}| = |\hat{B}| = 90°$. That is $ABCD$ is a rectangle.

Theorem 1.5.2

A quadrilateral $ABCD$ is a rhombus if and only if its diagonals are perpendicular bisectors.

Proof:

"$\Rightarrow$": Suppose that $ABCD$ is a rhombus. Then it is a parallelogram and hence its diagonals $\overline{AC}$ and $\overline{BD}$ bisect each other. We need to show that $\overline{AC} \perp \overline{BD}$. In the two triangles $\triangle ADO$ and $\triangle CDO$, we have:



1. $\overline{OD}$ is common.
2. $\overline{AD} \cong \overline{CD}$ (It is rhombus).
3. $\overline{AO} \cong \overline{CO}$ (It is parallelogram).

By SSS: $\triangle ADO \cong \triangle CDO$. Hence $A\hat{O}D \cong C\hat{O}D$ and both are right angles.

"$\Leftarrow$": Suppose that $ABCD$ is a quadrilateral with its diagonals are perpendicular bisector. Since $\overline{AC}$ and $\overline{BD}$ bisect each other, then $ABCD$ is a parallelogram. In the two triangles $\triangle ADO$ and $\triangle CDO$, we have:

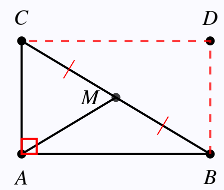
1. $\overline{OD}$ is common.
2. $A\hat{O}D \cong C\hat{O}D$ (given).
3. $\overline{AO} \cong \overline{CO}$ (given).

By SAS: $\triangle ADO \cong \triangle CDO$. Thus $\overline{AD} \cong \overline{CD}$ which implies that $ABCD$ is a rhombus.

Example 1.5.1

Show that the point M is equidistant from the vertices of the right tiangle.

Solution:



Let D be the point of intersection of the lines $\overleftrightarrow{CD}$ (parallel to $\overleftrightarrow{AB}$), and $\overleftrightarrow{BD}$ (parallel to $\overleftrightarrow{AC}$). By this construction, we get the parallelogram $ABDC$ which has a right angle $\hat{A}$. Thus we get the rectangle $ABDC$. Since it is a rectangle, its diagonals $\overline{AD}$ and $\overline{BC}$ bisect each other. That is, $\left|\overline{AM}\right| = \left|\overline{BM}\right| = \left|\overline{CM}\right|$.

Chapter

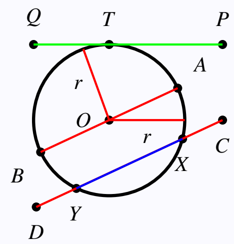
2 Circles

In this chapter we consider: Basic notions and definitions of circles. Circle theorems. Cyclic quadrilateral.

2.1 Notions and Definitions

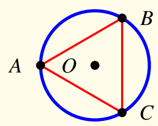
Definition 2.1.1

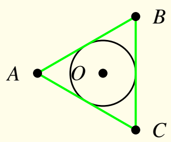
1. A **circle** is a set of points at a given distance (called **radius** "r") from a given point (called **center**). All **radii** of a circle are congruents.
2. A **chord** is a segment whose endpoints on a circle. Drawn as $\overline{XY}$.
3. A **secant** is a line that contains a chord. Drawn as $\overleftrightarrow{CD}$.
4. A **diameter** is a chord containing the center of a circle. Drawn as $\overline{AB}$.
5. A **tangent** is a line intersecting the circle in exactly one point called the **point of tangency**. Drawn as $\overleftrightarrow{PQ}$. The tangency point here is T.
6. We write $c(A,r)$ to denote a circle with radius r centered at point A. We also write $\odot A$ to denote a circle centered at point A.
7. Congruent circles $c(A,r) \cong c(B,r)$ are circles with congruent radii.



A polygon is **inscribed in a circle** and the circle is **circumscribed about the polygon** when each vertex of the polygon lies on the circle. In that case, the polygon is called **cyclic**.

If each side of a polygon is tangent to a circle, the polygon is said to be **circumscribed about the circle** and the circle is **inscribed in the polygon**.



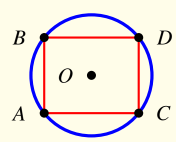


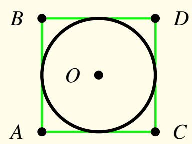
Inscribed polygons

circumscribed circles

circumscribed polygons

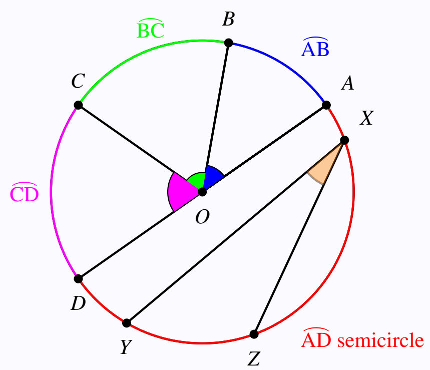
Inscribed circles





Definition 2.1.2

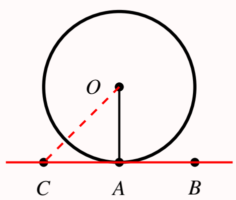
1. A **central angle** $A\hat{O}B$ of a circle is an angle whose vertex at the center. Examples of central angles: $A\hat{O}B$, $B\hat{O}C$, and $C\hat{O}D$.
2. A, B and the inbetween points of the circle form an **arc**, denoted $\overset{\frown}{\mathrm{AB}}$.
3. If A and B were the endpoints of a diameter, then the arc is called **semicircle**.
4. **Adjacent arcs** of a circle are arcs with exactly one common point. Arcs $\overset{\frown}{\mathrm{AB}}$ and $\overset{\frown}{\mathrm{BC}}$ are adjacent.
5. The measure of an arc is defined to be the measure of its central angle. $\left|\overset{\frown}{\mathrm{AB}}\right| = \left|A\hat{O}B\right|$.
6. Congruent arcs are arcs having the same measure.
7. An **inscribed angle** $Y\hat{X}Z$ is an angle whose vertex X is on the circle and whose sides contain chords $\overline{XY}$ and $\overline{XZ}$ of the circle. In that case, we say that angle $Y\hat{X}Z$ **intercept** the arc $\overset{\frown}{\mathrm{YZ}}$.



Theorem 2.1.1

If $\overleftrightarrow{AB}$ is a line and $\odot O$ is a circle. Then $\overleftrightarrow{AB}$ is tangent to $\odot O$ at A if and only if $\overline{AB} \perp \overline{AO}$.

Sketch:



Theorem 2.1.2: The Two Tangent Theorem

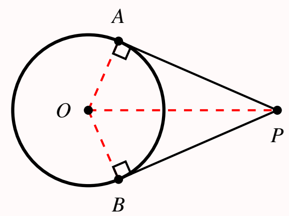
Tangents to a circle from a point P are congruent.

Proof:

In the two right triangles PAO and PBO ($|\hat{A}| = |\hat{B}| = 90^\circ$ since both points are tangency points), we have:

1. (leg) $\overline{AO} \cong \overline{BO}$ (radii).
2. (hypotenuse) $\overline{PO}$ is common.

By HL, we have $\triangle PAO \cong \triangle PBO$. Thus, $\overline{PA} \cong \overline{PB}$.



Theorem 2.1.3: The Arc Addition Theorem

The measure of the arc formed by two adjacent arcs equals the sum of the measure of these arcs.

Theorem 2.1.4

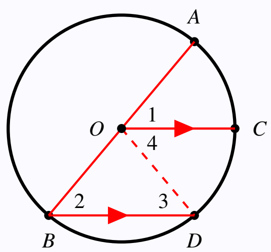
Two arcs are congruent if and only if their central angles are congruent.

Example 2.1.1

Given $\overline{AB}$ is a diameter of $\odot O$, and let $\overline{CO} \parallel \overline{BD}$. Show that $\overset{\frown}{AC} \cong \overset{\frown}{CD}$.

Solution:

If $\overline{CO} \parallel \overline{BD}$, then $\hat{1} \cong \hat{2}$ (corresponding angles). But then $\hat{2} \cong \hat{3}$ as the $\triangle OBD$ is isosceles triangle with congruent base angles. Then $\hat{3} \cong \hat{4}$ (alternate interior angles). That is $\hat{1} \cong \hat{4}$ which implies $\overset{\frown}{AC} \cong \overset{\frown}{BD}$.

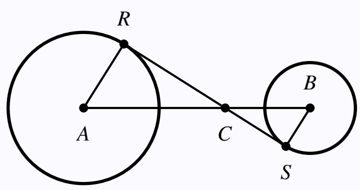


Example 2.1.2

Let $\overline{RS}$ be tangent to $\odot A$ and $\odot B$. Show that $\triangle ARC \sim \triangle BSC$.

Solution:

Clearly, $A\hat{C}R \cong B\hat{C}S$ (vertically opposite). Also $\hat{S} \cong \hat{R}$ (both are right angles). By S-AA, $\triangle ARC \cong \triangle BSC$.

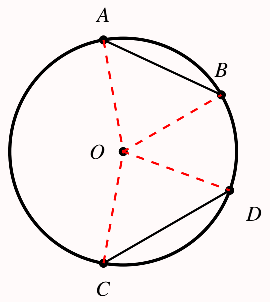


2.2 Arcs, Chords, and Angles of Circles

Theorem 2.2.1

Two chords are congruent if and only if their intercepted arcs are congruent.

Proof:



" $\Rightarrow$": Suppose that $\overline{AB} \cong \overline{CD}$. Then: $\triangle AOB$ and $\triangle COD$ have:

1. $\overline{AB} \cong \overline{CD}$ (given).
2. $\overline{AO} \cong \overline{CO}$ and $\overline{BO} \cong \overline{DO}$ (all of length radius r).

By SSS, we have $\triangle AOB \cong \triangle COD$. Hence $A\hat{O}B \cong C\hat{O}D$ which implies that $\overset{\frown}{AB} \cong \overset{\frown}{CD}$.

" $\Leftarrow$": Suppose that $\overset{\frown}{AB} \cong \overset{\frown}{CD}$. Then $A\hat{O}B \cong C\hat{O}D$. In $\triangle AOB$ and $\triangle COD$ we have:

1. $\overline{AO} \cong \overline{CO}$ and $\overline{BO} \cong \overline{DO}$ (all of length radius r).
2. $A\hat{O}B \cong C\hat{O}D$ (given).

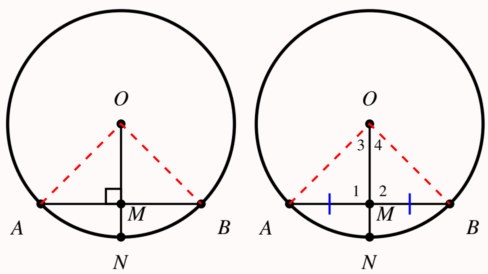
By SAS, we have $\triangle AOB \cong \triangle COD$. Hence $\overline{AB} \cong \overline{CD}$.

Theorem 2.2.2

Let $\overline{ON}$ be the segment joining the center O to a point N on the circle. Then: $\overline{ON} \perp \overline{AB}$ if and only if $\overline{ON}$ bisects $\overline{AB}$. In either case, $\overline{ON}$ bisects $\overset{\frown}{AB}$.

Given: $\odot O$; and $\overline{ON}$. Then $\overline{ON} \perp \overline{AB}$ iff $\overline{ON}$ bisects $\overline{AB}$. Moreover, $\overline{ON}$ bisects $\overset{\frown}{AB}$.

Proof:



" $\Rightarrow$": Suppose that $\overline{ON} \perp \overline{AB}$ intersecting in point M. In **right** triangles $\triangle OAM$ and $\triangle OBM$:

1. (hypotenuse) $\overline{OA} \cong \overline{OB}$ (both have length radius-r).
2. (leg) $\overline{OM}$ is common.

By HL, we have $\triangle OAM \cong \triangle OBM$. Hence, $\overline{AM} \cong \overline{BM}$. Also, $M\hat{O}A \cong M\hat{O}B$ which implies that $\overset{\frown}{AN} \cong \overset{\frown}{BN}$.

" $\Leftarrow$": Suppose that $\overline{ON}$ bisects $\overline{AB}$. Then, $\overline{AM} \cong \overline{BM}$. In triangles $\triangle OAM$ and $\triangle OBM$, we have

1. $\overline{AO} \cong \overline{BO}$ (both have length radius-r).

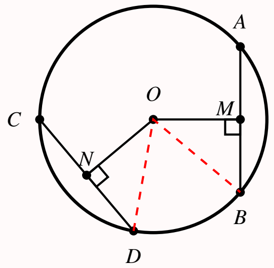
2. $\overline{AM} \cong \overline{BM}$ (given).
3. $\overline{OM}$ is common.

By SSS, we have $\triangle\, OAM \cong \triangle\, OBM$. Hence $\hat{1} \cong \hat{2}$ which implies that both angles are right. That is $\overline{ON} \perp \overline{AB}$. Moreover, $\hat{3} \cong \hat{4}$ which implies that $\overset{\frown}{\mathrm{AN}} \cong \overset{\frown}{\mathrm{BN}}$.

Theorem 2.2.3

Two chords are congruent if and only if they are equidistant from the center.

Proof:



" $\Rightarrow$": Suppose that $\overline{AB} \cong \overline{CD}$. Then $\overset{\frown}{\mathrm{AB}} \cong \overset{\frown}{\mathrm{CD}}$. Let $\overline{OM} \perp \overline{AB}$ and $\overline{ON} \perp \overline{CD}$. We now need to show that $\overline{OM} \cong \overline{ON}$. By Theorem 2.2.2, M and N are midpoints for $\overline{AB}$ and $\overline{CD}$. Then the **right** triangles $\triangle\, OBM$ and $\triangle\, ODN$ have:

1. (hypotenuse) $\overline{OB} \cong \overline{OD}$ (both have length radius-r).
2. (leg) $|\overline{ND}| = 1/2|\overline{CD}| = 1/2|\overline{AB}| = |\overline{MB}|$ (given: $|\overline{AB}| = |\overline{CD}|$).

By HL, we have $\triangle\, OBM \cong \triangle\, ODN$. Hence, $\overline{OM} \cong \overline{ON}$.

" $\Leftarrow$": Suppose that $\overline{OM} \cong \overline{ON}$ "$\overline{AB}$ and $\overline{CD}$ are equidistant". Then the **right** triangles $\triangle\, OBM$ and $\triangle\, ODN$ have:

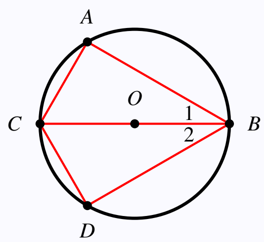
1. $\overline{OM} \cong \overline{ON}$ (given).
2. $\overline{OB} \cong \overline{OD}$ (both have length radius-r).
3. $\hat{M} \cong \hat{N}$ (both right angles).

By HL, we have $\triangle\, OBM \cong \triangle\, ODN$. Hence $\overline{BM} \cong \overline{DN}$. But since $\overline{OM}$ and $\overline{ON}$ are perpendicular to $\overline{AB}$ and $\overline{CD}$, Theorem 2.2.2 implies that M and N are midpoint of $\overline{AB}$ and $\overline{CD}$. Therefore, $\overline{AB} \cong \overline{CD}$.

Example 2.2.1

Let $\overset{\frown}{\mathrm{AB}} \cong \overset{\frown}{\mathrm{DB}}$. Show that $\hat{1} \cong \hat{2}$.

Solution:



Given $\overset{\frown}{\mathrm{AB}} \cong \overset{\frown}{\mathrm{BD}}$, we get $\overline{AB} \cong \overline{BD}$ by Theorem 2.2.1. Therefore, $|\overset{\frown}{\mathrm{AC}}| = 180^\circ - |\overset{\frown}{\mathrm{AB}}| = 180^\circ - |\overset{\frown}{\mathrm{BD}}| = |\overset{\frown}{\mathrm{CD}}|$. Thus $\overset{\frown}{\mathrm{AC}} \cong \overset{\frown}{\mathrm{CD}}$ which implies that $\overline{AC} \cong \overline{CD}$. By SSS, $\triangle\, ABC \cong \triangle\, DBC$ and hence $\hat{1} \cong \hat{2}$.

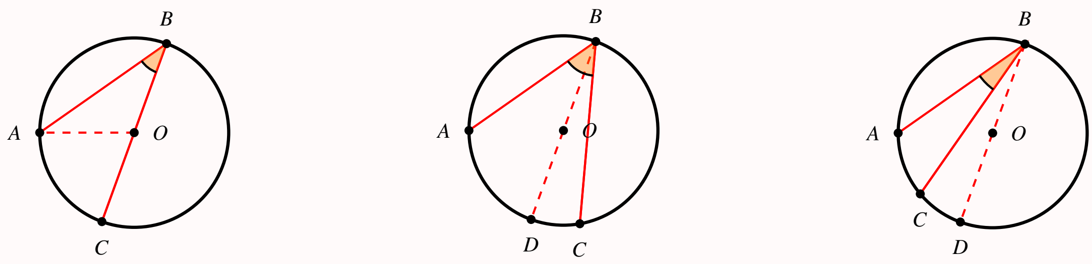
Theorem 2.2.4

The measure of an inscribed angle in a circle equals half of the measure of its intercepted arc.

Given: $\odot O$ and inscribed angle $A\hat{B}C$. Then: $\left|A\hat{B}C\right| = \frac{1}{2}\left|\overset{\frown}{AC}\right|$.

Proof:

We have three cases for such inscribed angle whether its chords passing through the center or not:



Case 1 **Case 2** **Case 3**

Case 1: Draw line $\overline{OA}$. Then $\triangle\ OAB$ is isosceles with $\left|B\hat{A}O\right| = \left|A\hat{B}O\right| = x$. Then $A\hat{O}C$ is an exterior angle to the triangle. That is $\left|A\hat{O}C\right| = 2x$. That is, $\left|\overset{\frown}{AC}\right| = 2x$. Therefore, $\left|A\hat{B}O\right| = x = \frac{1}{2}\left|\overset{\frown}{AC}\right|$.

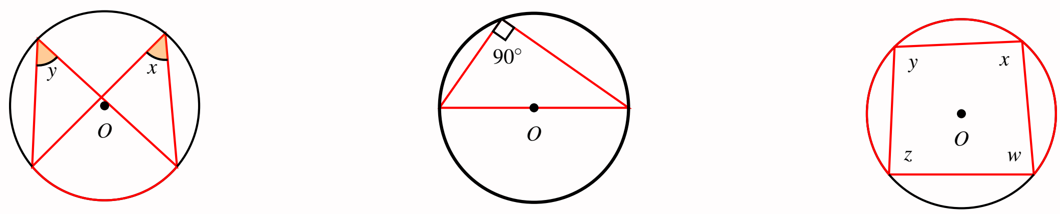
Case 2: Draw diameter $\overline{BD}$ passing through the center O. By case 1: $\left|C\hat{B}D\right| = \frac{1}{2}\left|\overset{\frown}{CD}\right|$ and $\left|D\hat{B}A\right| = \frac{1}{2}\left|\overset{\frown}{AD}\right|$. Thus $\left|A\hat{B}C\right| = \left|A\hat{B}D\right| + \left|D\hat{B}C\right| = \frac{1}{2}\left|\overset{\frown}{AD}\right| + \frac{1}{2}\left|\overset{\frown}{CD}\right| = \frac{1}{2}\left|\overset{\frown}{AC}\right|$.

Case 3: Draw diameter $\overline{BD}$ passing through the center O. By case 1: $\left|A\hat{B}C\right| + \left|C\hat{B}D\right| = \left|A\hat{B}D\right| = \frac{1}{2}\left|\overset{\frown}{AD}\right|$ and $\left|C\hat{B}D\right| = \frac{1}{2}\left|\overset{\frown}{CD}\right|$. Thus

$$\left|A\hat{B}C\right| = \left|A\hat{B}D\right| - \left|C\hat{B}D\right| = \frac{1}{2}\left|\overset{\frown}{AD}\right| - \frac{1}{2}\left|\overset{\frown}{CD}\right| = \frac{1}{2}\left(\left|\overset{\frown}{AC}\right| + \left|\overset{\frown}{CD}\right|\right) - \frac{1}{2}\left|\overset{\frown}{CD}\right| = \frac{1}{2}\left|\overset{\frown}{AC}\right|.$$

Corollary 2.2.1: Based on Theorem 2.2.4

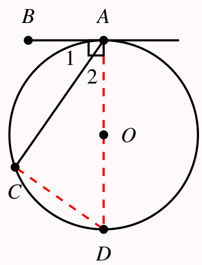
1. Any two inscribed angles intercepting the same arc are congruent.
2. An angle inscribed in a semicircle is a right angle.
3. An inscribed quadrilateral in a circle have opposite supplementary angles.



1: $\hat{x} \cong \hat{y}$. **2:** right angle. **3:** $x + z = 180° = y + w$.

Theorem 2.2.5

The measure of an angle formed by a chord and a tangent is half as the measure of the intercepted arc. That is: in the diagram $|B\hat{A}C| = \frac{1}{2}|\overset{\frown}{AC}|$.

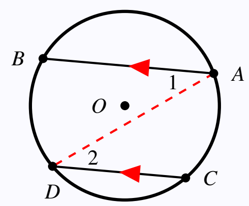


Proof:

Draw $\overline{AD}$ passing through O and join C and D. By Theorem 2.1.1 $\overline{AB} \perp \overline{AD}$, and hence $|B\hat{A}D| = 90°$. That is $|\hat{1}| + |\hat{2}| = 90°$.
By Corollary 2.2.1, we have $|\hat{C}| = 90°$. Also, $|\hat{2}| + |\hat{D}| = \frac{1}{2}180° = 90°$. Thus, $\hat{1} \cong \hat{D}$, but $|\hat{D}| = \frac{1}{2}|\overset{\frown}{AC}| = |\hat{1}|$.

Example 2.2.2

If two chords of a circle are parallel, then the two arcs between the chords are congruent.

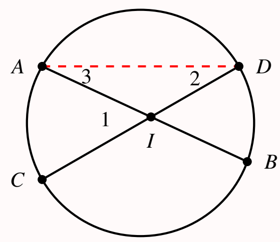


Solution:

Since $\overline{AB} \parallel \overline{CD}$, we have $\hat{1} \cong \hat{2}$ (alternate interior angles). Thus, $|\overset{\frown}{AC}| = 2|\hat{2}| = 2|\hat{1}| = |\overset{\frown}{BD}|$. That is $\overset{\frown}{AC} \cong \overset{\frown}{BD}$.

Theorem 2.2.6

The measure of an angle formed by intersected chords in a circle equals to half the sum of the intercepted arcs. That is: in the diagram $|\hat{1}| = \frac{1}{2}(|\overset{\frown}{AC}| + |\overset{\frown}{BD}|)$.



Proof:

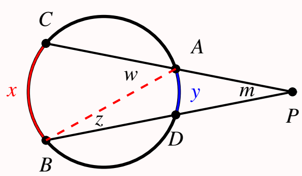
Draw $\overline{AD}$. Then $|\hat{1}| = |\hat{2}| + |\hat{3}|$ as $\hat{1}$ is an exterior angle to $\triangle IAD$. But $|\hat{2}| = \frac{1}{2}|\overset{\frown}{AC}|$ and $|\hat{3}| = \frac{1}{2}|\overset{\frown}{BD}|$. Hence,

$$|\hat{1}| = \frac{1}{2}|\overset{\frown}{AC}| + \frac{1}{2}|\overset{\frown}{BD}| = \frac{1}{2}(|\overset{\frown}{AC}| + |\overset{\frown}{BD}|).$$

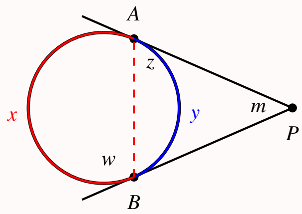
Theorem 2.2.7

The measure of an angle formed by (1) two secants, (2) two tangents, or (3) a secant and a tangent drawn from a point outside a circle equals half the difference of the measure of its intercepted arcs. That is, in all cases (of the diagram), show that $m = \frac{1}{2}(x - y)$.

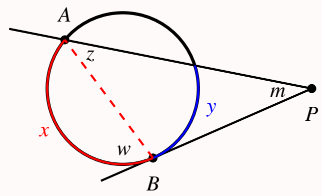
Proof:



1: two secants.



2: two tangents.



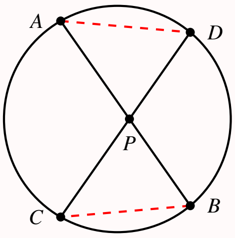
3: a secant and a tangent.

In any case, we have (exterior angle of $\triangle ABP$) $w = m + z$. Thus, $m = w - z$. But $z = \frac{1}{2}y$ and $w = \frac{1}{2}x$. That is, $m = \frac{1}{2}x - \frac{1}{2}y = \frac{1}{2}(x - y)$.

Theorem 2.2.8

When two chords intersect in a circle, the product of the segments of one chord equals the product of the segments of the other chord.

Given: $\overline{AB}$ intersects $\overline{CD}$ at P. Then: $|\overline{AP}| \cdot |\overline{PB}| = |\overline{CP}| \cdot |\overline{PD}|$.



Proof:

Draw segments $\overline{AD}$ and $\overline{BC}$. In triangles $\triangle\, PAD$ and $\triangle\, PCB$, we have

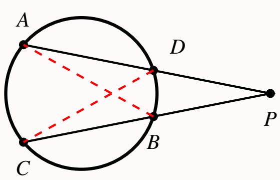
1. $\hat{A} \cong \hat{C}$ and $\hat{B} \cong \hat{D}$ (share same intercepted arcs).
2. $A\hat{P}D \cong C\hat{P}B$ (vertically opposite).

By S-AA: $\triangle\, PAD \sim \triangle\, PCB$. That is $\dfrac{|\overline{AP}|}{|\overline{CP}|} = \dfrac{|\overline{PD}|}{|\overline{PB}|}$ or similarly, $|\overline{AP}| \cdot |\overline{PB}| = |\overline{CP}| \cdot |\overline{PD}|$.

Theorem 2.2.9

When two secants intersect a circle, the product of the segments of one secant equals the product of the segments of the other secant.

Given: $\overline{PA}$ and $\overline{PC}$ intersects a circle at D and B. Then: $|\overline{PA}| \cdot |\overline{PD}| = |\overline{PC}| \cdot |\overline{PB}|$.



Proof:

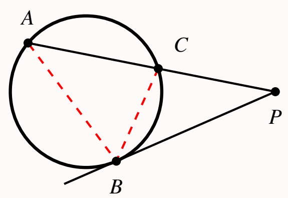
Draw segments $\overline{AB}$ and $\overline{CD}$. In triangles $\triangle PAB$ and $\triangle PCD$, we have

1. $\hat{A} \cong \hat{C}$ (share same intercepted arc $\widehat{\mathrm{BD}}$).
2. $\hat{P}$ is common.

By S-AA: $\triangle PAB \sim \triangle PCD$. That is $\dfrac{|\overline{PA}|}{|\overline{PC}|} = \dfrac{|\overline{PB}|}{|\overline{PD}|}$ or similarly, $|\overline{PA}| \cdot |\overline{PD}| = |\overline{PC}| \cdot |\overline{PB}|$.

Theorem 2.2.10

If $\overline{PB}$ is a tangent and $\overline{PA}$ is a secant drawn from a point P outside a circle, then $|\overline{PB}|^2 = |\overline{PA}| \cdot |\overline{PC}|$.



Proof:

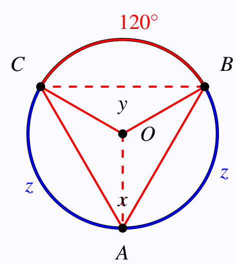
Draw segments $\overline{AB}$ and $\overline{BC}$. In triangles $\triangle PAB$ and $\triangle PBC$, we have

1. $P\hat{A}B \cong P\hat{B}C$ (share same intercepted arc $\widehat{\mathrm{BC}}$).
2. $\hat{P}$ is common.

By S-AA: $\triangle PAB \sim \triangle PBC$. That is $\dfrac{|\overline{PA}|}{|\overline{PB}|} = \dfrac{|\overline{PB}|}{|\overline{PC}|}$ or similarly, $|\overline{PA}| \cdot |\overline{PC}| = |\overline{PB}|^2$.

Example 2.2.3

In the diagram, let $|\widehat{\mathrm{BC}}| = 120°$. Also assume that $\widehat{\mathrm{AB}} \cong \widehat{\mathrm{AC}}$.



1. Find x, y and z.
2. Show that $\triangle ABO \cong \triangle ACO$.
3. Find the distance between B and C in terms of $|\overline{AB}|$ and $|\overline{AC}|$.

Solution:

1. Clearly, $x = \frac{1}{2}|\widehat{\mathrm{BC}}| = 60°$, and $y = |\widehat{\mathrm{BC}}| = 120°$. Thus, (completing the circle 360°), we have $2z = 360° - 120° = 240°$ and hence $z = 120°$.
2. in the triangles $\triangle ABO$ and $\triangle ACO$, we have:
   (a) $\overline{BO} \cong \overline{CO}$ (both are radii).
   (b) $\overline{AB} \cong \overline{AC}$ (since $\widehat{\mathrm{AB}} \cong \widehat{\mathrm{AC}}$).
   (c) $\overline{AO}$ is common.

   By SSS, $\triangle ABO \cong \triangle ACO$.

3. As $\overset{\frown}{\mathrm{BC}} \cong \overset{\frown}{\mathrm{AB}}$, we have $\left|\overline{BC}\right| = \left|\overline{AB}\right| = \left|\overline{AC}\right|$.

Chapter

3 Locus

3.1 The locus

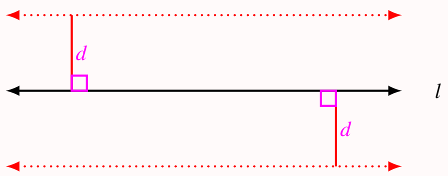
Definition 3.1.1

A **locus** (plural: **loci**) (Latin word for "location") is a set of points that satisfy one or more conditions.

Theorem 3.1.1

Given a line $\overleftrightarrow{l}$. The locus of points that at distance d from $\overleftrightarrow{l}$ is the points of two parallel lines at distance d.

The condition: All points at distance d from $\overleftrightarrow{l}$. The locus of such points are forming two parallel lines to $\overleftrightarrow{l}$.



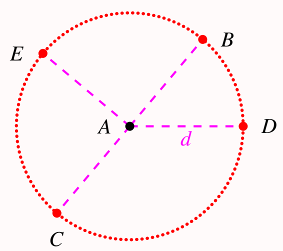
Theorem 3.1.2

Let A be a fixed point in the plane. The locus of points that at distance r from A are the points of the circle centered at A with radius r.

Note that any point B lies on the locus must satisfy the condition $\left|\overline{AB}\right| = r$.

Also, any point B on the circle must satisfy $\left|\overline{AB}\right| = r$.

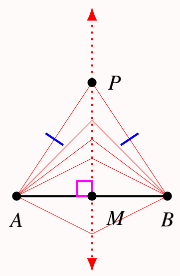
Hence the locus is a circle centered at A with radius r.



Theorem 3.1.3

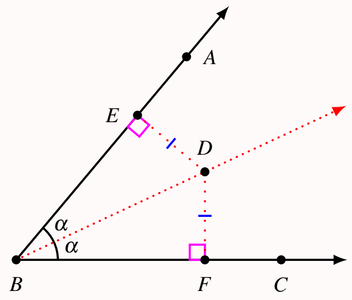
Given two fixed points A and B, the locus of points equidistant from A and B is the perpendicular bisector of $\overline{AB}$. Such a line is sometimes called **mediatrix**.

Let M be the midpoint of $\overline{AB}$. Then any point P lies on the perpendicular bisector if and only if it is equidistant from the endpoints (points A and B). That is, the locus of points that are equidistant from fixed points A and B are the points forming the perpendicular bisector of $\overline{AB}$. See Theorem 1.1.2.



Theorem 3.1.4

Given an angel $A\hat{B}C$, the locus of points equidistant from the sides of $A\hat{B}C$ (namely, $\overrightarrow{AB}$ and $\overrightarrow{BC}$) is the angle bisector.



Proof:

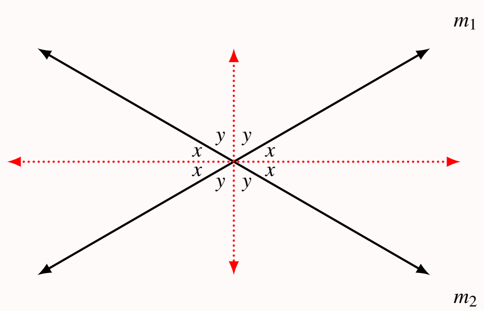
Here is a proof of "a point is on the angle bisector iff it is equidistant from its sides". This is a restate of Theorem 1.1.3.

Assume **first** that a point D is on the angle bisector of $A\hat{B}C$. In the triangles $\triangle\, EBD$ and $\triangle\, FDB$, we have $E\hat{B}D \cong F\hat{B}D$ (assumption). Also, $D\hat{E}B \cong D\hat{F}B$ for both are right angles. Since $\overline{BD}$ is common in both triangles, then by AAS, $\triangle\, EBD \cong \triangle\, FBD$. That is $\overline{ED} \cong \overline{FD}$ and the point D (which is on the locus) is equidistant from the sides.

Next, assume that the point D is equidistant from $\overrightarrow{AC}$ and $\overrightarrow{BC}$. Then, in triangles $\triangle\, EBD$ and $\triangle\, FBD$ we have $D\hat{E}B \cong D\hat{F}B$ (both are right angles). Also, $\overline{ED} \cong \overline{FD}$ (by assumption). By HL, we have $\triangle\, EBD \cong \triangle\, FBD$. Therefore, $E\hat{B}D \cong F\hat{B}D$ and hence $\overrightarrow{BD}$ is a bisector for the angle $\hat{B}$.

Theorem 3.1.5

The locus of points equidistant from two intersecting lines $\overleftrightarrow{m_1}$ and $\overleftrightarrow{m_2}$ is the pair of lines bisecting the angles formed by $\overleftrightarrow{m_1}$ and $\overleftrightarrow{m_2}$.

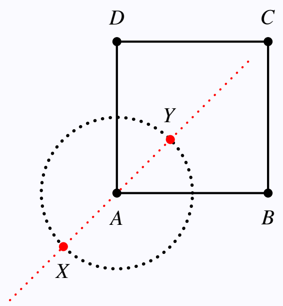


Example 3.1.1

Given a square $ABCD$ with sides r cm. Construct the locus of points which are $\frac{1}{2}r$ cm from A and equidistant from $\overline{AB}$ and $\overline{AC}$.

Solution:

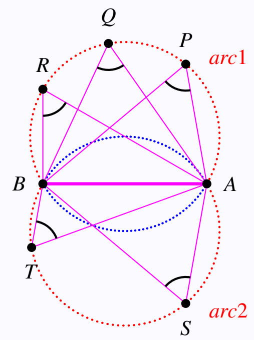
Note that the points that are equidistant from sides $\overline{AB}$ and $\overline{AC}$ are the points on the angle bisector of $B\hat{A}C$. Moreover, the points that are at distance $\frac{1}{2}r$ from A are the points on a circle centered at A with radius $\frac{1}{2}r$.



Therefore, the locus of points that are equidistant from sides $\overline{AB}$ and $\overline{AC}$ and that are at distance $\frac{1}{2}r$ from A are the two points X and Y.

Example 3.1.2

Let A and B be two fixed points. If P moves in the plane such that $\left|A\hat{P}B\right|$ is a constant, find the locus of such points.



Solution:

The locus of points P that keep the same angle measure $\left|A\hat{P}B\right| = k$ consists of two arcs (arc 1 and arc 2) of circles of the same radius symmetric through $\overline{AB}$ (points A and B do not belong to the locus).

Assume that P lies on a circle with some radius such that the smaller arc $\widehat{\mathrm{AB}}$ has a measure $\left|\widehat{\mathrm{AB}}\right| = 2k$. Hence all points on bigger arc of $\widehat{\mathrm{AB}}$ form an inscribed angle with measure $\left|A\hat{P}B\right| = \frac{1}{2}2k = k$. Note that P can be on either circles the one on top or on the bottom.

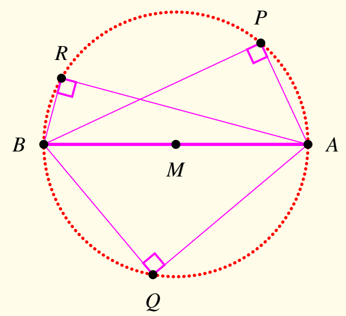
Assume that $\left|A\hat{P}B\right| = k$. Then the angle is inscribed in a circle with P is a vertex on the circle facing arc $\widehat{\mathrm{AB}}$ with $\left|\widehat{\mathrm{AB}}\right| = 2k$.

Remark 3.1.1

Note that if the constant angle in Example 3.1.2 was 90° (right angle).

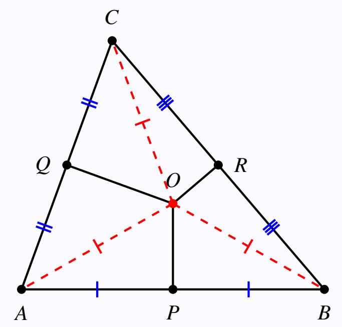
Then $\overline{AB}$ would be a diameter of a circle (the locus) centered at the midpoint of $\overline{AB}$ and with radius $\frac{1}{2}\left|\overline{AB}\right|$.

That is the locus of points preserving the right angle lie on the circle and facing an arc of measure 180°. That is a semicircle.



Example 3.1.3

Find the locus of points that are equidistant from three fixed points (non collinear) A, B, and C.



Solution:

Note that the points that are equidistant from A and B lie on the perpendicular bisector of $\overline{AB}$, namely $\overline{PO}$. Also, the points that are equidistant from A and C are the points on $\overline{QO}$. The points that are equidistant from B and C are on the perpendicular bisector $\overline{RO}$.

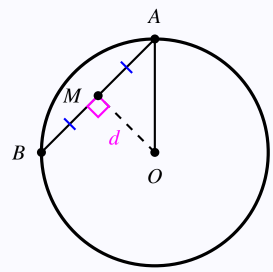
Therefore, the equidistant point from all of the three points must lie on the intersection of the three perpendicular bisectors of $\overline{AB}$, $\overline{AC}$, and $\overline{BC}$.

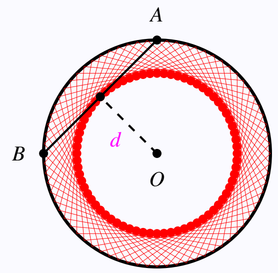
Then the locus is only one point O satisfying the locus condition $|\overline{AO}| = |\overline{BO}| = |\overline{CO}|$ which is the circumcenter of $\triangle ABC$.

Example 3.1.4

Given a circle $c(O,r)$ and a chord $\overline{AB}$ moving such that $|\overline{AB}|$ is a constant. Find the locus of the midpoints of $\overline{AB}$.

Solution:





The locus is a circle centered at O of radius $d < r$, where $d = |\overline{OM}|$.

Reasoning: We will show that $|\overline{OM}| = d$ is a constant distance. That is when M moves around along with its chord, a circle forming the locus is created. Note that $|\overline{OA}| = r$ is a constant. Also, M is a midpoint of $\overline{AB}$ and hence $|\overline{AM}| = \frac{1}{2}|\overline{AB}|$ is a constant as well. But by Pythagorean Theorem, we have $|\overline{OM}|^2 = |\overline{OA}|^2 - |\overline{AM}|^2$ which is also a constant.

Therefore, the locus of midpoints M is a circle $c(O,d)$.

Chapter

4 Transformations

4.1 Isometries

Definition 4.1.1

A **transformation** is a bijective (ono-to-one and onto) mapping of E^2 (the plane) onto itself.

That is if $\mathbf{T}$ is a transformation, then for every point P in the plane, there is a unique point Q such that $\mathbf{T}(P) = Q$. Conversely, for every point Q there is a unique point P such that $\mathbf{T}(P) = Q$.

In that case, we say that Q is the **image** of P, and that P is the **preimage** of Q.

Definition 4.1.2

If $\mathbf{T}$ is a transformation satisfying the property that "if P, Q, R are three collinear points, then $\mathbf{T}(P), \mathbf{T}(Q), \mathbf{T}(R)$ are collinear", then $\mathbf{T}$ is called a **collineation**.

Definition 4.1.3

Let $\mathbf{T}$ and $\mathbf{S}$ be any two transformations, then

- The **identity** transformation is the transformation $\mathbf{I}$ defined by $\mathbf{I}(P) = P$ for every point P.
- The inverse transformation of $\mathbf{T}$, denoted $\mathbf{T}^{-1}$, is defined by $\mathbf{T}^{-1}(Q) = P$ iff $\mathbf{T}(P) = Q$.
- The **composition** (or **product**) of $\mathbf{T}$ and $\mathbf{S}$ (which is also a transformation) is denoted $\mathbf{S} \circ \mathbf{T}$ (or as a product $\mathbf{ST}$) and is defined by $\mathbf{S} \circ \mathbf{T}(P) = \mathbf{S}(\mathbf{T}(P))$.

Note that $\mathbf{S}$ is the inverse of $\mathbf{T}$ iff $\mathbf{ST} = \mathbf{I} = \mathbf{TS}$.

Definition 4.1.4

An **isometry** (iso-metry: equal-distance) is a transformation that maps every segment to a congruent segment. That is, an isometry preserves distance. In notation: $\mathbf{T}$ is an isometry iff $\mathbf{T}\left(\overline{AB}\right) = \overline{A'B'}$ with $\overline{AB} \cong \overline{A'B'}$.

Theorem 4.1.1

The product (composition) of two isometries is an isometry.

Proof:

Let $\mathbf{T}$ and $\mathbf{S}$ be two isometries. Then for any A and B, $\mathbf{T}\left(\overline{AB}\right) = \overline{A'B'}$ and $\mathbf{S}\left(\overline{A'B'}\right) = \overline{A''B''}$, where $\overline{AB} \cong \overline{A'B'}$ ($\mathbf{T}$ is isometry) and $\overline{A'B'} \cong \overline{A''B''}$ ($\mathbf{S}$ is isometry).

Therefore, $\mathbf{ST}\left(\overline{AB}\right) = \mathbf{S}\left(\mathbf{T}\left(\overline{AB}\right)\right) = \mathbf{S}\left(\overline{A'B'}\right) = \overline{A''B''}$, with $\overline{AB} \cong \overline{A''B''}$.

Theorem 4.1.2

The identity transformation is an isometry.

Theorem 4.1.3

Let $\mathbf{T}$ be any isometry. Then

- $\mathbf{T}^{-1}$ is also an isometry.
- If $\mathbf{T}$ fixes A and B, then $\mathbf{T}$ fixes $\overleftrightarrow{AB}$.
- If $\mathbf{T}$ fixes any three noncollinear points, then $\mathbf{T} = \mathbf{I}$.

Theorem 4.1.4

Two isometries agree on three noncollinear points are identical.

Proof:

Let $\mathbf{T}$ and $\mathbf{S}$ be two isometries such that $\mathbf{T}(A) = \mathbf{S}(A)$, $\mathbf{T}(B) = \mathbf{S}(B)$, $\mathbf{T}(C) = \mathbf{S}(C)$, for noncollinear points A, B, C. Then $\mathbf{S}^{-1}\mathbf{T}(A) = A$, $\mathbf{S}^{-1}\mathbf{T}(B) = B$, $\mathbf{S}^{-1}\mathbf{T}(C) = C$. That is $\mathbf{S}^{-1}\mathbf{T} = \mathbf{I}$. Hence, $\mathbf{T} = \mathbf{S}$.

Theorem 4.1.5

An isometry is a $(\star)$ collineation that preserves (a) betweenness; (b) midpoints; (c) segments; (d) rays; (e) triangles; (f) angles; (g) angle measure; (h) perpendicularity; (i) parallelism;

Proof:

Let $\mathbf{T}$ be any isometry. Suppose that A, B, C are any three points in the plane, and let $\mathbf{T}(A) = P, \mathbf{T}(B) = Q, \mathbf{T}(C) = R$. Then:

(a) betweenness: If $\left|\overline{AB}\right| + \left|\overline{BC}\right| = \left|\overline{AC}\right|$, then as $\mathbf{T}$ isometry, $\left|\overline{PQ}\right| + \left|\overline{QR}\right| = \left|\overline{PR}\right|$. Hence, if B is between A and C, then Q is between P and R. That is, $\mathbf{T}$ preserves betweenness.

(b) midpoints: If B is the midpoint of $\overline{AC}$, then $\left|\overline{AB}\right| = \left|\overline{BC}\right|$. By part (a), we get $\left|\overline{PQ}\right| = \left|\overline{QR}\right|$ and hence Q is the midpoint of $\overline{PR}$. $\mathbf{T}$ preserves midpoints.

(c) segments: This is clear by the definition of isometry $\mathbf{T}$, we have $\mathbf{T}\left(\overline{AB}\right) = \overline{PQ}$ with $\overline{AB} \cong \overline{PQ}$. $\mathbf{T}$ preserves segments.

(d) rays: Note that $\overrightarrow{AB}$ is the union of $\overline{AB}$ and all points C such that B is between A and C. Thus, $\mathbf{T}\left(\overrightarrow{AB}\right)$ is the union of $\overline{PQ}$ and all points R such that Q is between P and R. So, $\mathbf{T}\left(\overrightarrow{AB}\right) = \overrightarrow{PQ}$. That is, $\overrightarrow{AB} \cong \overrightarrow{PQ}$. Thus, $\mathbf{T}$ preserves rays.

($\star$) Since $\overleftrightarrow{AB}$ is the union of $\overrightarrow{AB}$ and $\overrightarrow{BA}$, we have $\mathbf{T}\left(\overleftrightarrow{AB}\right)$ is the union of $\overrightarrow{PQ}$ and $\overrightarrow{QP}$, which is $\overleftrightarrow{PQ}$. $\mathbf{T}$ preserves lines and hence $\mathbf{T}$ is a collineation.

(e) triangles: If A, B, C are noncollinear, then $\left|\overline{AB}\right| + \left|\overline{BC}\right| > \left|\overline{AC}\right|$ and hence $\left|\overline{PQ}\right| + \left|\overline{QR}\right| > \left|\overline{PR}\right|$ (noncollinear). Moreover, $\triangle\, ABC$ is the union of the segments $\overline{AB}, \overline{BC}, \overline{AC}$. By part (c), $\mathbf{T}$ preserves segments and hence $\triangle\, ABC \cong \triangle\, PQR$ by SSS. That is, $\mathbf{T}$ preserves triangles.

(f) angles: part (e) implies that $A\hat{B}C \cong P\hat{Q}R$ since $\mathbf{T}\left(A\hat{B}C\right) = P\hat{Q}R$. $\mathbf{T}$ preserves angles.

(g) angle measure: by part (f), $\left|A\hat{B}C\right| = \left|P\hat{Q}R\right|$. $\mathbf{T}$ preserves angle measures.

(h) perpendicularity: by part (g), if $\overline{AB} \perp \overline{BC}$, then $\left|A\hat{B}C\right| = 90^\circ$. Thus, $\left|P\hat{Q}R\right| = 90^\circ$ and hence $\overline{PQ} \perp \overline{QR}$. $\mathbf{T}$ preserves perpendicularity.

(i) parallelism: isometry preserves angles and hence it preserves parallelism.

4.2 Reflections

Definition 4.2.1

A **reflection** in a line m (called **mirror**), denoted $\mathbf{R}_m$, is the transformation defined by

$$P \mapsto \begin{cases} P, & \text{if } P \in \overleftrightarrow{m}; \\ Q, & \text{otherwise, and } \overleftrightarrow{m} \text{ is the perpendicular bisector of } \overline{PQ}. \end{cases}$$

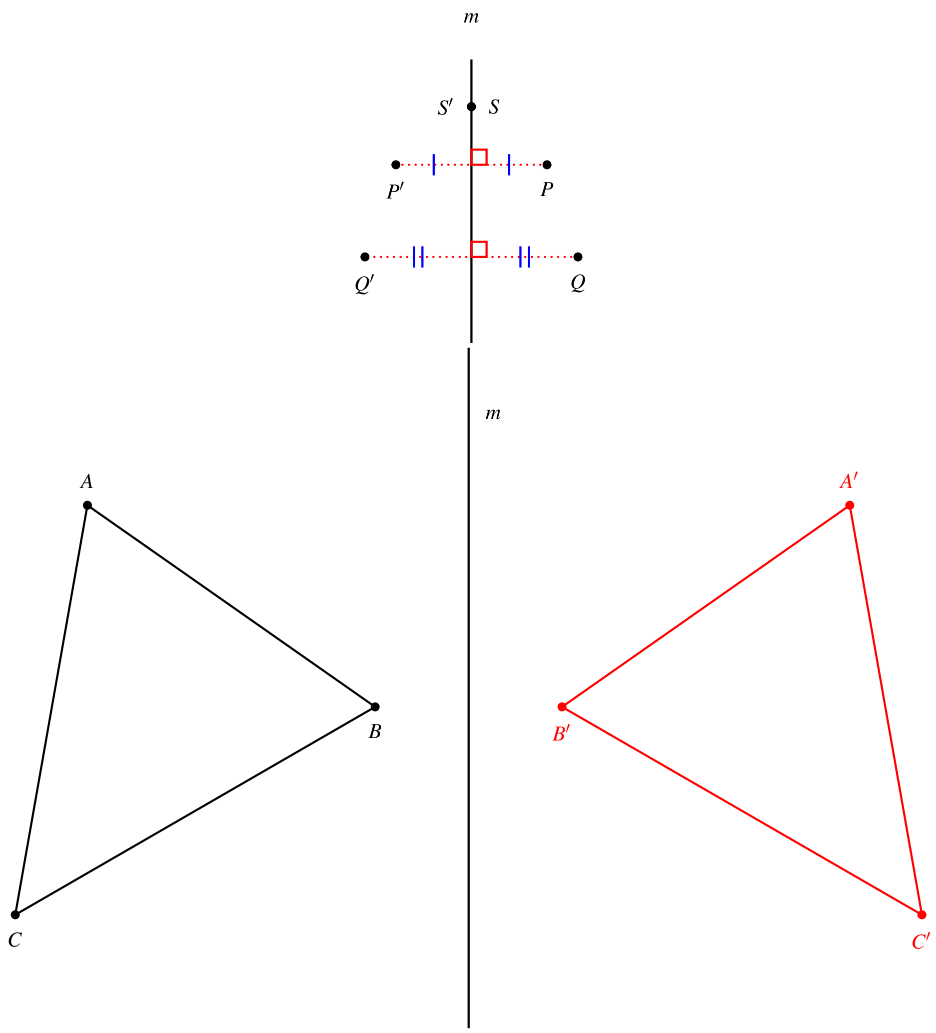


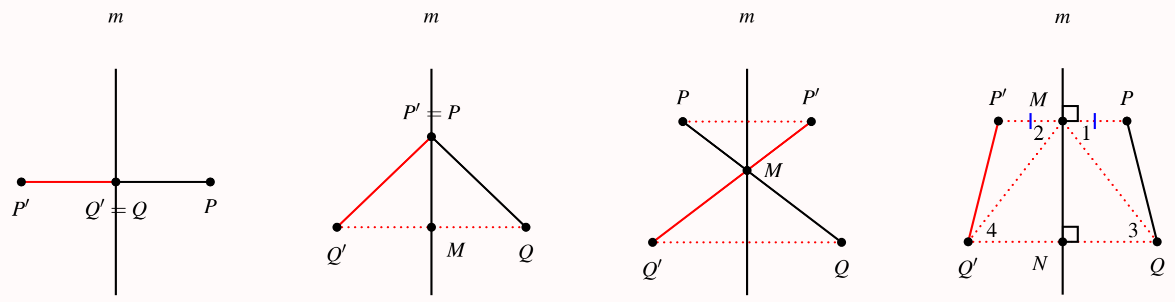
Figure 4.1: Reflection in a line m.

Theorem 4.2.1

A reflection is an isometry.

Proof:

Here we must show that $\left|\overline{PQ}\right| = \left|\overline{P'Q'}\right|$ for all choices of P and Q. Here are some possible cases we will prove (all reflections are made in the line m. That is, $\mathbf{R}_m$):



Case 1 **Case 2** **Case 3** **Case 4**

Figure 4.2: Reflection of some points in line m.

Case 1: $P \notin m$ and hence line m is perpendicular bisector of $\overline{PP'}$. Since $Q \in m$, $Q' \in m$. Therefore, Q is the midpoint of $\overline{PP'}$ and hence $\left|\overline{PQ}\right| = \left|\overline{P'Q'}\right|$.

Case 2: $Q \notin m$ and hence m is the perpendicular bisector of $\overline{QQ'}$. Let $M \in m$ be the midpoint of $\overline{QQ'}$. In right triangles $\triangle\, PQM$ and $\triangle\, P'Q'M$, we have

(a) $\overline{PM}$ is common.

(b) $\overline{QM} \cong \overline{Q'M}$ (m is a bisector).

(c) $Q\hat{M}P \cong Q'\hat{M}P'$ (m is perpendicular on $\overline{QQ'}$).

By SAS, $\triangle\, PQM \cong \triangle\, P'Q'M$, and hence $\overline{PQ} \cong \overline{P'Q'}$.

Case 3: Let M be the intersection point of m with $\overline{PQ}$. By **Case 2**, we have $\overline{PM} \cong \overline{P'M}$ and $\overline{QM} \cong \overline{Q'M}$. Therefore, $\overline{PQ} \cong \overline{P'Q'}$.

Case 4: By **Case 2**, $\overline{MQ} \cong \overline{MQ'}$ and hence $\triangle\, MQQ'$ is isosceles with $\hat{3} \cong \hat{4}$. Since m is perpendicular to both $\overleftrightarrow{PP'}$ and $\overleftrightarrow{QQ'}$, we obtain that $\overleftrightarrow{PP'} \parallel \overleftrightarrow{QQ'}$. Therefore, $\hat{1} \cong \hat{3} \cong \hat{4} \cong \hat{2}$. So, in $\triangle\, PQM$ and $\triangle\, P'Q'M$, we have:

(a) $\overline{PM} \cong \overline{P'M}$ ($\overleftrightarrow{m}$ is perpendicular bisector of $\overline{PP'}$).

(b) $\overline{QM} \cong \overline{Q'M}$ (**Case 2**).

(c) $\hat{1} \cong \hat{2}$ (proved).

By SAS, we have $\triangle\, PQM \cong \triangle\, P'Q'M$ and hence $\overline{PQ} \cong \overline{P'Q'}$.

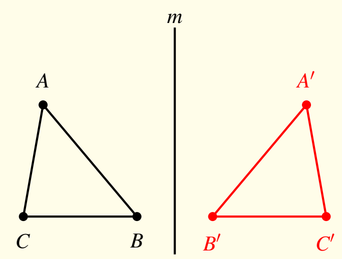
Definition 4.2.2

Given an isometry $\mathbf{T}$, then $\mathbf{T}$ is

- a **direct isometry** if it preserves the orientation. That is, the order of lettering in the figure and the image are the same: either both clockwise or both counterclockwise.
- an **opposite isometry** if it does not preserve the orientation. That is, the order of lettering is reversed.
- a **periodic** if $\mathbf{T}^n = \mathbf{I}$ for some integer n. In that case, we say that $\mathbf{T}$ is periodic with **period** n.

Remark 4.2.1

A product of two reflections in the same line is the identity. That is, $\mathbf{R}_m^2 = \mathbf{I}$. If m is any line then $\mathbf{R}_m^{-1} = \mathbf{R}_m$. Therefore, the reflection is periodic with period 2.



Reflections are **opposite isometries** not preserving the orientation and reversing the lettering.

Here is a table for the composition of two isometries with respect to direct or opposite property:

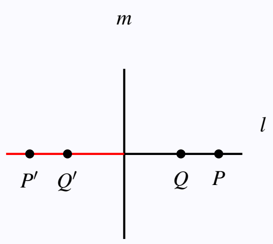
$\circ$	direct	opposite
direct	direct	opposite
opposite	opposite	direct

We note that, given any isometry $\mathbf{T}$, we can show that $\mathbf{T}$ is a reflection if it is opposite isometry fixing at least a point.

Isometry	**Direct / Opposite**	**Fixed Points**
Translation	Direct	NO
Rotation	Direct	YES
Reflection	Opposite	YES
Glide Reflection	Opposite	NO

Example 4.2.1

Let m and l be two perpendicular lines in the plane. Find the reflection of l in m, that is $\mathbf{R}_m(l)$.

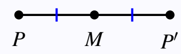


Solution:

Let P and Q be two points on l. Then $\mathbf{R}_m(P) = P' \in l$ and $\mathbf{R}_m(Q) = Q' \in l$. That is $\mathbf{R}_m(l) = \overleftrightarrow{P'Q'} = \overleftrightarrow{l}$.

Example 4.2.2

Let $\mathbf{T}$ be an opposite isometry of period 2. Show that $\mathbf{T}$ is a reflection.



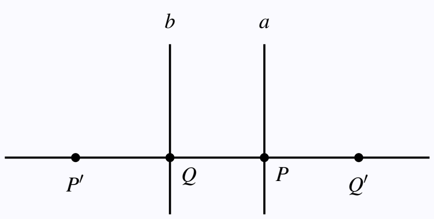
Solution:

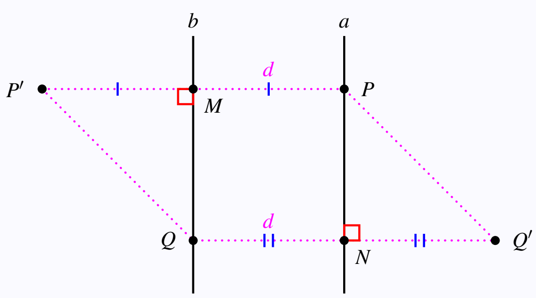
We need to show that $\mathbf{T}$ is an opposite isometry and fixing a point. $\mathbf{T}$ is of period 2 implies that $\mathbf{T}^2 = \mathbf{I}$. Take any point P. Then $\mathbf{T}(P) = P'$ and $\mathbf{T}(\mathbf{T}(P)) = P$ (since $\mathbf{T}^2 = \mathbf{I}$). That is, $\mathbf{T}(P') = P$. If M is the midpoint of $\overline{PP'}$, then $\mathbf{T}(M) = M$ and hence M is a fixed point. That is $\mathbf{T}$ is a reflection whose mirror is the perpendicular bisector of $\overline{PP'}$ passing through M.

Example 4.2.3

Let $\mathbf{R}_a$ and $\mathbf{R}_b$ be two reflections in two parallel lines a and b, respectively. Let $P \in a$ and $Q \in b$ be points with $\mathbf{R}_b(P) = P'$ and $\mathbf{R}_a(Q) = Q'$. Show that P, P', Q, Q' are either collinear or the vertices of a parallelogram.

Solution:





Case 1: Assume that P and Q are on the same line $\overleftrightarrow{PQ}$. Then $\overline{PQ} \perp a$ and $\overline{PQ} \perp b$. Also $\mathbf{R}_b(P) = P' \in \overleftrightarrow{PQ}$ and $\mathbf{R}_a(Q) = Q' \in \overleftrightarrow{PQ}$. Thus, P, P', Q, Q' are collinear.

Case 2: Assume that P and Q are not on the same line. Then $\overline{PQ} \not\perp a$ or b. But $\mathbf{R}_b(P) = P'$. Then $\overline{PP'} \perp b$ and $\left|\overline{PP'}\right| = 2d$ (where d is the distance between a and b). Also, $\mathbf{R}_a(Q) = Q'$ which implies $\overline{QQ'} \perp a$ and $\left|\overline{QQ'}\right| = 2d$. Therefore, $\overline{PP'} \parallel \overline{QQ'}$ with $\overline{PP'} \cong \overline{QQ'}$. Therefore, $PP'QQ'$ is a parallelogram.

4.3 Rotations

Definition 4.3.1

A **rotation** about point O through angle x°, denoted $\mathcal{R}_{O,x}$, is the transformation defined by

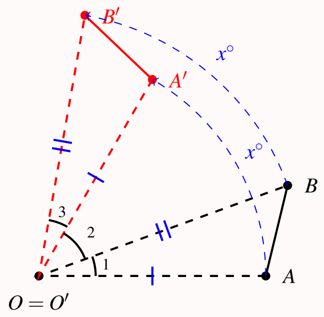
$$P \mapsto \begin{cases} P, & \text{if } P = O; \\ Q, & \text{otherwise, and } \left|\overline{OP}\right| = \left|\overline{OQ}\right| \text{ and } \left|P\hat{O}Q\right| = x^\circ. \end{cases}$$

If in addition $x = 180^\circ$, then we say that $\mathcal{R}_{O,180}$ is a **half-turn** and denote it as $\mathcal{H}_O$. As a result $\mathcal{H}_O$ is periodic with period 2.

Theorem 4.3.1

A rotation is an isometry.

Solution:



Consider a rotation $\mathcal{R}_{O,x}$ about some point O through x°. Let A and B be points in the plane with $\mathcal{R}_{O,x}(A) = A'$ and $\mathcal{R}_{O,x}(B) = B'$. Then, we need to show that $\left|\overline{AB}\right| = \left|\overline{A'B'}\right|$. In $\triangle AOB$ and $\triangle A'OB'$, we have:

1. $\left|\overline{OA}\right| = \left|\overline{OA'}\right|$ and $\left|\overline{OB}\right| = \left|\overline{OB'}\right|$ (definition of rotation).
2. $\left|\hat{1}\right| = x - \left|\hat{2}\right| = \left|\hat{3}\right|$ (look at diagram).

By SAS, $\triangle AOB \cong \triangle A'OB'$. That is $\left|\overline{AB}\right| = \left|\overline{A'B'}\right|$.

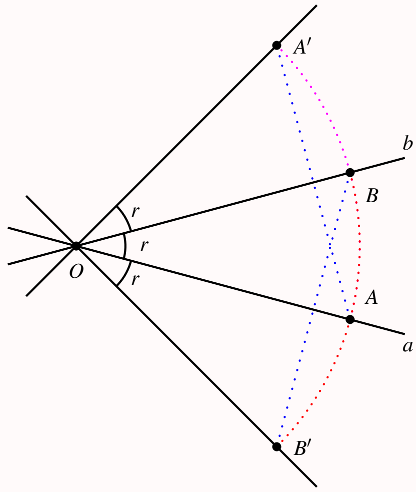
Remark 4.3.1

Let $\mathcal{R}_{O,x}$ be a rotation about point O through x°. Then:

- A rotation is a direct isometry.
- The only invariant point is O. It is called the center of the rotation.
- By the definition of a half-turn we have $\mathcal{H}_O^2 = \mathbf{I}$.
- The composition of two rotations about the same center is a rotation: $\mathcal{R}_{O,x} \circ \mathcal{R}_{O,y} = \mathcal{R}_{O,x+y}$.
- The inverse of a rotation is a rotation. That is, $\mathcal{R}_{O,x}^{-1} = \mathcal{R}_{O,-x}$.
- A half-turn about O is a composition of two reflections in perpendicular lines, say l and m, intersecing in O. That is, $\mathcal{H}_O = \mathbf{R}_l \circ \mathbf{R}_m = \mathbf{R}_m \circ \mathbf{R}_l$. See Example 4.2.1.

Theorem 4.3.2

A composition of two reflections in two intersecting lines is simply a rotation about the intersection point through doubled the angle between the two lines.



Proof:

Let a and b be two lines intersecting in a point O with the angle inbetween measures r. Let A be a point in a distinct from O and let B be the intersection of line b with the circle $\odot O$ centered at O with radius $\left|\overline{OA}\right|$. Then, $\left|A\hat{O}B\right| = r$ and $\overleftrightarrow{OB} = b$. Let $A' = \mathcal{R}_{O,2r}(A)$. Then, A' is on the circle $\odot O$ and b is perpendicular bisector of $\overline{AA'}$ (properties of circles). So, $A' = \mathbf{R}_b(A)$. Now let $B' = \mathbf{R}_a(B)$. Then a is perpendicular bisector of $\overline{BB'}$ (definition of reflections), and the directed angle $\left|B'\hat{O}B\right| = 2r$. Thus we have:

- $\mathbf{R}_b(\mathbf{R}_a(O)) = \mathbf{R}_b(O) = O = \mathcal{R}_{O,2r}(O)$.
- $\mathbf{R}_b(\mathbf{R}_a(B')) = \mathbf{R}_b(B) = B = \mathcal{R}_{O,2r}(B')$.
- $\mathbf{R}_b(\mathbf{R}_a(A)) = \mathbf{R}_b(A) = A' = \mathcal{R}_{O,2r}(A)$.

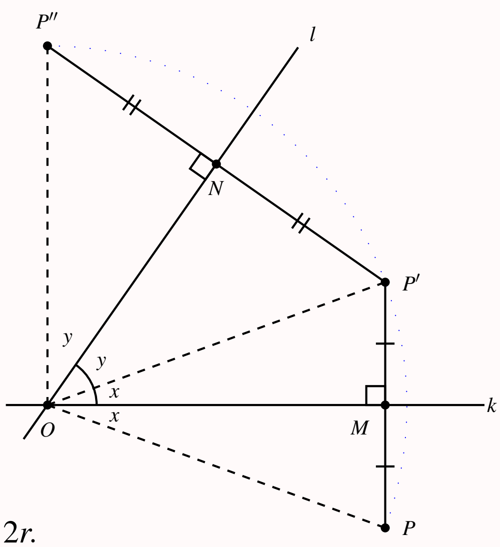
Since O, A, B' are three noncollinear points, Theorem 4.1.4 implies that $\mathbf{R}_b \circ \mathbf{R}_a = \mathcal{R}_{O,2r}$.

Proof:

Here is another proof. Assume that lines k and line l intersect at point O with a directed angle from k to l equals to $r = (x+y)$.

Let P be a point so that $\mathbf{R}_k(P) = P'$ and $\mathbf{R}_l(P') = P''$.

We have $\triangle\, OPM \cong \triangle\, OP'M$ and $\triangle\, OP'N \cong \triangle\, OP''N$, by SAS. Hence we have



$$(1) \quad \cdots \quad \left|\overline{OP}\right| = \left|\overline{OP'}\right| = \left|\overline{OP''}\right|.$$

Moreover, the (directed) angle from $\overline{OP}$ to $\overline{OP''}$ is

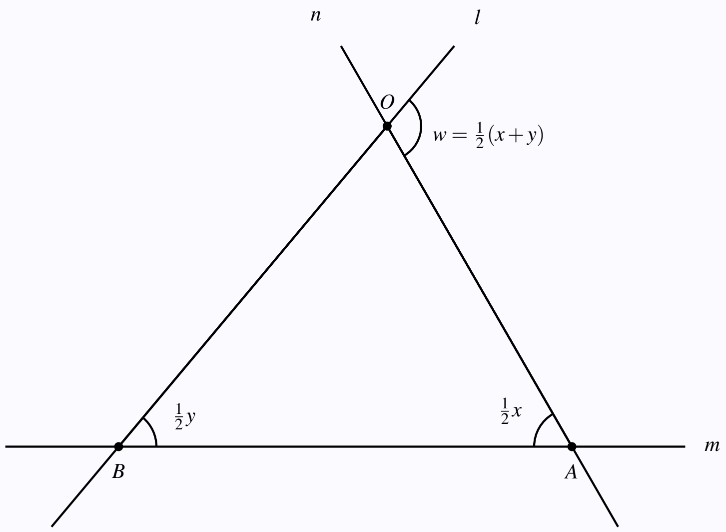
$$(2) \quad \cdots \quad 2(x+y) = 2r.$$

Therefore, from (1) and (2) we have $\mathbf{R}_l \circ \mathbf{R}_k = \mathcal{R}_{O,2r}$.

Example 4.3.1

Let $\mathcal{R}_{A,x}$ and $\mathcal{R}_{B,y}$ be two rotations with distinct centers A and B. Assuming the $x+y$ is not a multiple of $360°$, find $\mathcal{R}_{O,\theta} = \mathcal{R}_{B,y} \circ \mathcal{R}_{A,x}$.

Solution:



Recall that any rotation is a composition of two reflections in two intersected lines with the angle between lines is half the angle of the rotation.

Let n and m be two lines intersected in A with an angle from n to m equals to $\frac{x}{2}$, and l be the line intersects with m in B with an angle from m to l equals to $\frac{y}{2}$. Since $x+y$ is not a multiple of 360, lines l and n intersects as in the diagram, say at point O, with an angle from n to l equals $w = \frac{1}{2}(x+y)$ as it is an exterior angle of the $\triangle ABO$.

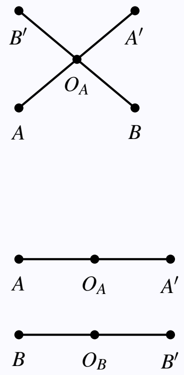
Let $\mathcal{R}_{A,x} = \mathbf{R}_m \circ \mathbf{R}_n$, and $\mathcal{R}_{B,y} = \mathbf{R}_l \circ \mathbf{R}_m$. Then

$$\mathcal{R}_{O,\theta} = \mathcal{R}_{B,y} \circ \mathcal{R}_{A,x} = \mathbf{R}_l \circ \mathbf{R}_m \circ \mathbf{R}_m \circ \mathbf{R}_n = \mathbf{R}_l \circ \mathbf{R}_n.$$

Recall that $\mathbf{R}_m \circ \mathbf{R}_m = I$. That is the rotation $\mathcal{R}_{O,\theta}$ is infact a composition of two reflections in lines a_1 and b_2 intersecting in O (the new center of rotation) through angle $\theta = 2w = (x+y)$, note that w is the angle between a_1 and b_2.

Example 4.3.2

Show that every isometry of period 2 is either a reflection or a half-turn.



Solution:

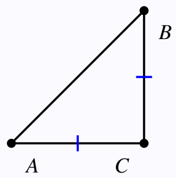
Let $\mathbf{T}$ be an isometry of period 2. Hence $\mathbf{T}^2 = \mathbf{I}$. For any point A, we have $\mathbf{T}(A) = A'$ and $\mathbf{T}(A') = A$ so that if O_A is the midpoint of $\overline{AA'}$, we have $\mathbf{T}(O_A) = O_A$.

Case 1: Any other point we choose, say B, we get $\mathbf{T}(O_B) = O_B = O_A$. Then $\mathbf{T}$ is a half-turn.

Case 2: Any other point we choose, say B, we get $\mathbf{T}(O_B) = O_B \neq O_A$. Then $\mathbf{T}$ is a reflection.

Example 4.3.3

Let $\triangle ABC$ be a triangle with the vertices labelled clockwise such that $\left|\overline{AC}\right| = \left|\overline{BC}\right|$ and $\left|A\hat{C}B\right| = 90°$. Let $\mathbf{R}_{\overleftrightarrow{AB}}$ be the reflection in the line $\overleftrightarrow{AB}$, $\mathbf{R}_{\overleftrightarrow{AC}}$ be the reflection in the line $\overleftrightarrow{AC}$, and $\mathcal{R}_{B,90°}$ be the rotation by $90°$ counterclockwise around B. Identify the composition $\mathcal{R}_{B,90°} \circ \mathbf{R}_{\overleftrightarrow{AB}} \circ \mathbf{R}_{\overleftrightarrow{AC}}$.



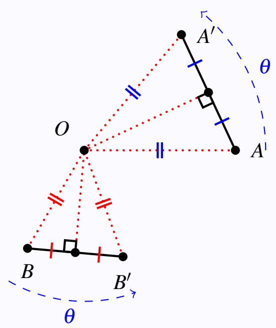
Solution:

Note that $\mathbf{R}_{\overleftrightarrow{AB}} \circ \mathbf{R}_{\overleftrightarrow{AC}}$ is simply the rotation $\mathcal{R}_{A,90°}$. That is,

$$\mathcal{R}_{B,90°} \circ \left(\mathbf{R}_{\overleftrightarrow{AB}} \circ \mathbf{R}_{\overleftrightarrow{AC}}\right) = \mathcal{R}_{B,90°} \circ \mathcal{R}_{A,90°} = \left(\mathbf{R}_{\overleftrightarrow{BC}} \circ \mathbf{R}_{\overleftrightarrow{AB}}\right) \circ \left(\mathbf{R}_{\overleftrightarrow{AB}} \circ \mathbf{R}_{\overleftrightarrow{AC}}\right) = \mathbf{R}_{\overleftrightarrow{BC}} \circ \mathbf{R}_{\overleftrightarrow{AC}} = \mathcal{R}_{C,180°}.$$

Example 4.3.4

Given two points A and B in the plane, and their respective images A' and B' under a rotation $\mathcal{R}_{O,\theta}$. Construct (Find) the center of rotation O.



Solution:

Clearly, join A with A' and B with B' and then take the perpendicular bisector for $\overline{AA'}$ and $\overline{BB'}$. The center of rotation then is the intersection point of the two perpendicular bisectors.

In the case of $\theta = 180°$ and that A, B, O are collinear, then the center would be the midpoint of $\overline{AA'}$ which is exactly the midpoint of $\overline{BB'}$.

4.4 Translations

Definition 4.4.1

A **translation** (or a **glide**) is a transformation that glide all points of the plane in the same direction with the same distance.

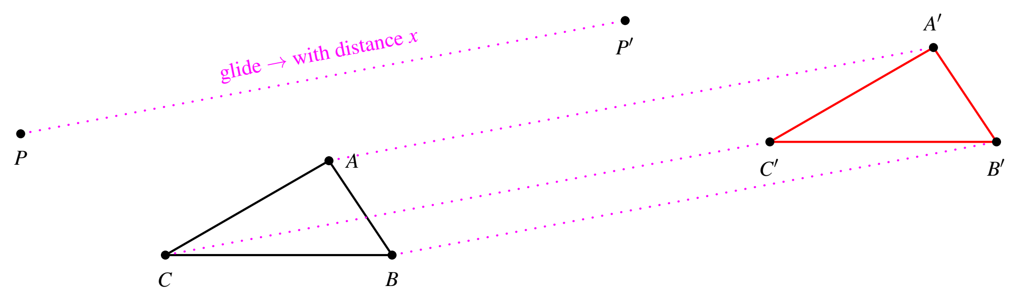


Figure 4.3: Translation with distance x.

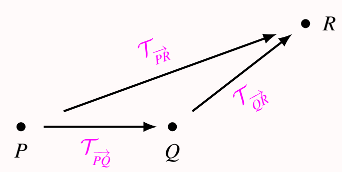
Remark 4.4.1

- A translation is a direct isometry.
- A nonidentity translation fixes no points in the plane.
- Given points A and B, there is a unique translation moving A to B. Thus, we we write $\mathcal{T}_{\overrightarrow{AB}}$ to denote the translation mapping A to B.

Theorem 4.4.1

A composition of two translation is a translation

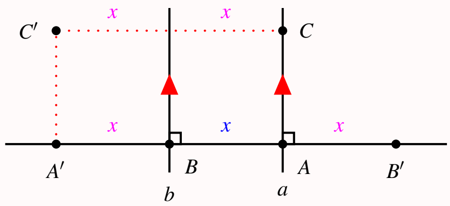
Solution:



Let $\mathcal{T}_{\overrightarrow{AB}}$, $\mathcal{T}_{\overrightarrow{CD}}$ be any two translations. Assume that for any point P, $\mathcal{T}_{\overrightarrow{AB}}(P) = Q$ and $\mathcal{T}_{\overrightarrow{CD}}(Q) = R$. Then, $\mathcal{T}_{\overrightarrow{CD}} \circ \mathcal{T}_{\overrightarrow{AB}}(P) = \mathcal{T}_{\overrightarrow{CD}}\left(\mathcal{T}_{\overrightarrow{AB}}(P)\right) = \mathcal{T}_{\overrightarrow{CD}}(Q) = R$. That is $\mathcal{T}_{\overrightarrow{CD}} \circ \mathcal{T}_{\overrightarrow{AB}} = \mathcal{T}_{\overrightarrow{PR}}$.

Theorem 4.4.2

A composition of two reflections in two parallel lines (with distance x between the lines) is a translation (with distance $2x$).



Solution:

Let a and b be two parallel lines and the distance between a and b is x. Let $\overleftrightarrow{AB}$ be perpendicular to both lines a and b with $A \in a$ and $B \in b$. Let C be a point on a distinct from A. Let $A' = \mathbf{R}_b(A)$ and $C' = \mathcal{T}_{\overrightarrow{AA'}}(C)$. Then, clearly $\mathcal{T}_{\overrightarrow{AA'}} = \mathcal{T}_{\overrightarrow{CC'}}$ and the glide distance is $2x$. Since b is the perpendicular bisector of $\overline{CC'}$, we have $\mathbf{R}_b(C) = C'$.

If $B' = \mathbf{R}_a(B)$, then A is the midpoint of $\overline{BB'}$ and also B is the midpoint of $\overline{AA'}$. Hence, $\mathcal{T}_{\overrightarrow{B'B}} = \mathcal{T}_{\overrightarrow{AA'}}$ with the same distance $2x$. Therefore, we have:

- $\mathbf{R}_b(\mathbf{R}_a(B')) = \mathbf{R}_b(B) = B = \mathcal{T}_{\overrightarrow{B'B}} = \mathcal{T}_{\overrightarrow{AA'}}$.
- $\mathbf{R}_b(\mathbf{R}_a(C)) = \mathbf{R}_b(C) = C' = \mathcal{T}_{\overrightarrow{CC'}} = \mathcal{T}_{\overrightarrow{AA'}}$.
- $\mathbf{R}_b(\mathbf{R}_a(A)) = \mathbf{R}_b(A) = A' = \mathcal{T}_{\overrightarrow{AA'}}$.

As A, B', C are three noncollinear points, Theorem 4.1.4 implies that $\mathbf{R}_b \circ \mathbf{R}_a = \mathcal{T}_{\overrightarrow{AA'}} = \mathcal{T}^2_{\overrightarrow{AB}}$.

Theorem 4.4.3

Every direct isometry of the plane is either a rotation or a translation.

Solution:

Let $\mathbf{T}$ be any direct (=opposite x opposite) isometry. Then $\mathbf{T} = \mathbf{R}_a\mathbf{R}_b$ a product of two reflections in lines a and b. If a is parallel to b, then $\mathbf{T}$ is a translation by Theorem 4.4.2. Otherwise it is a rotation by Theorem 4.3.2.

Theorem 4.4.4

Every translation is the product of two half-turns.

Solution:

Let $\mathcal{T}_{\overrightarrow{AB}}$ be any translation. Then $\mathcal{T}_{\overrightarrow{AB}}$ can be written as a product of two reflections in parallel lines a and b. That is, $\mathcal{T}_{\overrightarrow{AB}} = \mathbf{R}_a\mathbf{R}_b$. Let c be a line perpendicular to a and b in points O_1 and O_2. Then,

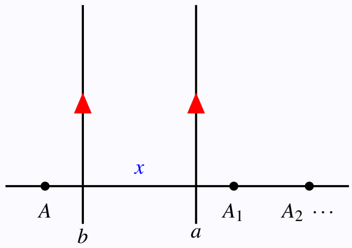
$\mathcal{H}_{O_1} = \mathbf{R}_a\mathbf{R}_c$ and $\mathcal{H}_{O_2} = \mathbf{R}_c\mathbf{R}_b$. Therefore,

$$\mathcal{T}_{\overrightarrow{AB}} = \mathbf{R}_a\mathbf{R}_b = \mathbf{R}_a\mathbf{I}\mathbf{R}_b = \mathbf{R}_a\mathbf{R}_c\mathbf{R}_c\mathbf{R}_b = \mathcal{H}_{O_1}\mathcal{H}_{O_2}.$$

Example 4.4.1

Show that the only periodic translation is the identity.

Solution:



Assume that $\mathcal{T}_{\overrightarrow{PQ}}$ is any translation. So it can be composed of two reflections in two parallel lines a and b with distance x. If $A_1 = \mathcal{T}_{\overrightarrow{PQ}}(A)$, then $\left|\overline{AA_1}\right| = 2x$. Let $A_2 = \mathcal{T}^2_{\overrightarrow{PQ}}(A) = \mathcal{T}_{\overrightarrow{PQ}}(A_1)$ and with $\left|\overline{A_2A_1}\right| = 2x$ and hence $\left|\overline{AA_2}\right| = 4x$. If n is the period of $\mathcal{T}_{\overrightarrow{PQ}}$, then $\mathcal{T}^n_{\overrightarrow{PQ}}(A) = A_n = A$ (assuming it is periodic with period n). Therefore, $\left|\overline{AA_n}\right| = 2nx = \left|\overline{AA}\right| = 0$. Hence $x = 0$ so the lines a and b are infact the same line. That is, the translation is the identity.

Example 4.4.2

Let $\mathcal{T}_{\overrightarrow{PQ}}$ be a translation taking P to Q at distance $2x$. Show that for any points $A \neq B$, if A, B, C, D form a quadrilateral, then it is a parallelogram, where $C = \mathcal{T}_{\overrightarrow{PQ}}(A)$ and $D = \mathcal{T}_{\overrightarrow{PQ}}(B)$.

Solution:

Let $\mathbf{R}_a, \mathbf{R}_b$ be two reflections so that $\mathcal{T}_{\overrightarrow{PQ}} = \mathbf{R}_b \circ \mathbf{R}_a$ with the distance between the lines is x. Thus, $\mathbf{R}_b \circ \mathbf{R}_a(A) = C$ with $\left|\overline{AC}\right| = 2x$ and $\mathbf{R}_b \circ \mathbf{R}_a(B) = D$ with $\left|\overline{BD}\right| = 2x$. Note that $a \perp \overline{AC}$ and also $a \perp \overline{BD}$ and hence $\overline{AC} \parallel \overline{BD}$. Therefore, $ABCD$ is a parallelogram.

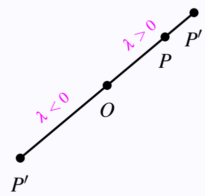
Chapter 5

Homothecy and Similarity

5.1 Homothecy

Definition 5.1.1

Let λ be a nonzero scalar. A **homothecy** (or **homothety**, or **dilation**), denoted $\mathcal{D}_{O,\lambda}$, is the transformation that maps O to itself and for any other point P,



$$P \mapsto \begin{cases} P' \in \overrightarrow{OP}, & \text{if } \lambda > 0; \\ P' \in \overrightarrow{PO}, & \text{if } \lambda < 0; \end{cases}$$

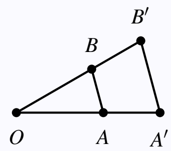
such that $\left|\overline{OP'}\right| = |\lambda| \, \left|\overline{OP}\right|$. The point O and the scalar λ are called the center of and the ratio of the homothecy, respectively.

Remark 5.1.1

- A homothecy is called **expansion** (**stretching**) if it ratio $|\lambda| > 1$, and it is called **contraction** (or **reduction**) if $|\lambda| < 1$.
- A homothecy maps a figure to a similar figure. It has exactly one fixed point in the plane.
- A homothecy $\mathcal{D}_{O,1}$ is the identity mapping $\mathbf{I}$, and a homothecy $\mathcal{D}_{O,-1}$ is the (reversed) identity mapping $-\mathbf{I}$.

Example 5.1.1

For a noncollinear points A, B, O, show that $\mathcal{D}_{O,\lambda}\left(\overline{AB}\right) = \overline{A'B'}$ implies $\dfrac{\left|\overline{AB'}\right|}{\left|\overline{AB}\right|} = |\lambda|$.



Solution:

Simply show that $\triangle\, OAB \sim \triangle\, OA'B'$ to get the result.

A homothecy is not an isometry in general. $\left|\overline{OA'}\right| = |\lambda|\ |\overline{OA}| \neq \left|\overline{OA'}\right|$ if $\lambda \neq |1|$.

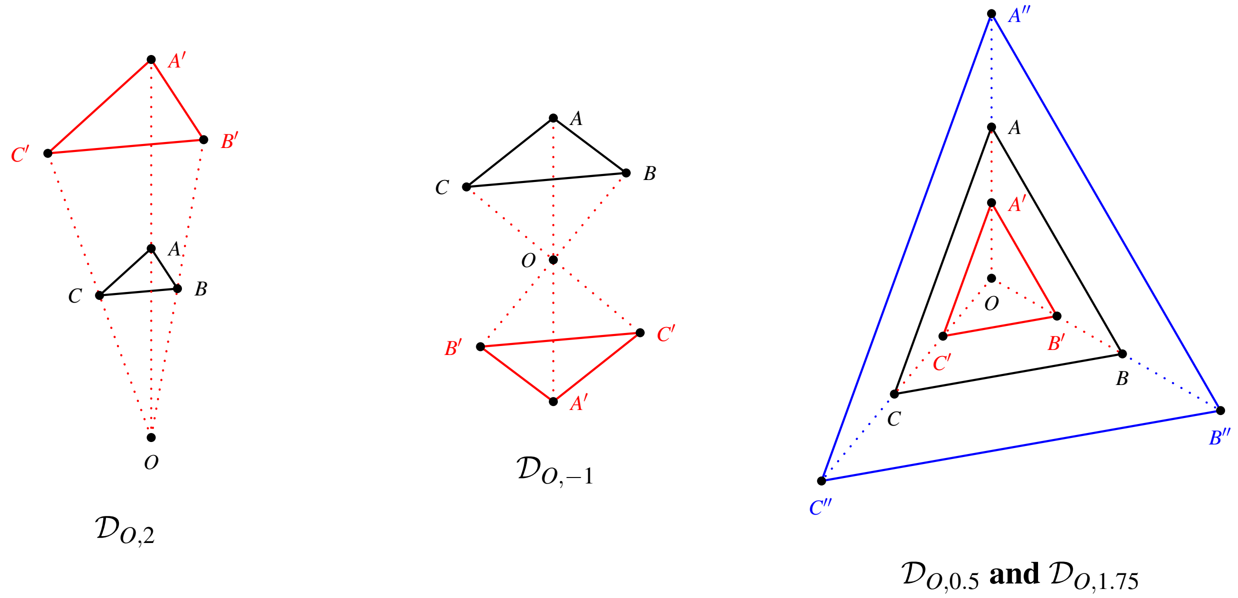


Figure 5.1: Some example of homothecy $\mathcal{D}_{O,\lambda}$ for different position of O and different values of λ.

Theorem 5.1.1

A homothecy maps any triangle to a similar triangle. Consequently, homothecies preserve angles.

Proof:

Let $\mathcal{D}_{O,\lambda}(\triangle\, ABC) = \triangle\, A'B'C'$ (see Figure 5.1). Clearly, by S-SAS, we have $\triangle\, OAB \sim \triangle\, OA'B'$, $\triangle\, OAC \sim \triangle\, OA'C'$, and $\triangle\, OBC \sim \triangle\, OB'C'$. Therefore, $\frac{\left|\overline{A'B'}\right|}{\left|\overline{AB}\right|} = \frac{\left|\overline{A'C'}\right|}{\left|\overline{AC}\right|} = \frac{\left|\overline{B'C'}\right|}{\left|\overline{BC}\right|} = |\lambda|$. By S-SSS, we have $\triangle\, ABC \sim \triangle\, A'B'C'$. Therefore, $\hat{A} \cong \hat{A}' = \hat{B} \cong \hat{B}' = \hat{C} \cong \hat{C}'$.

Example 5.1.2

Use the definition of a homothecy to show that a homothecy is a collineation.

Proof:

Let A, B, C be three collinear points in the plane, so that $|\overline{AC}| = |\overline{AB}| + |\overline{BC}|$. Then,

$$\left|\overline{A'C'}\right| = |\lambda|\,|\overline{AC}| = |\lambda|\left(|\overline{AB}| + |\overline{BC}|\right) = |\lambda|\ |\overline{AB}| + |\lambda|\ |\overline{BC}| = \left|\overline{A'B'}\right| + \left|\overline{B'C'}\right|.$$

That is $A'B'C'$ are collinear.

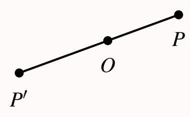
Theorem 5.1.2

A homothecy maps a line l to a parallel line l'.

Proof:

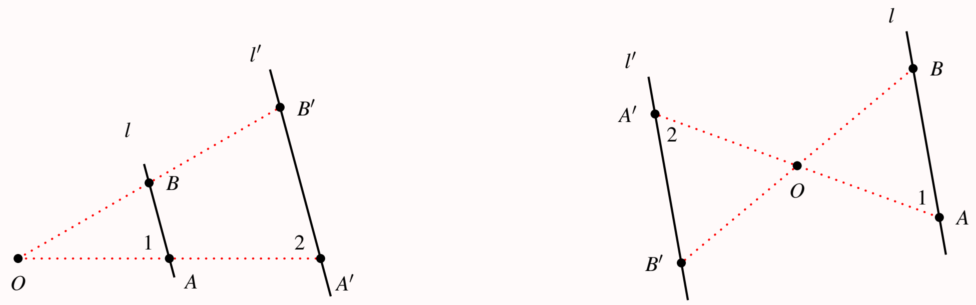
Let l be a line and $\mathcal{D}_{O,\lambda}(l) = l'$. Then:

Case 1: Assume that $O \in l$. Then by the definition $\mathcal{D}_{O,\lambda}$ maps each point to another point in the same line. That is $l = l'$.



Case 1

Case 2: Assume that $O \notin l$.



Case 2

Two triangles (see the diagram) $\triangle\, OAB$ and $\triangle\, OA'B'$ are similar (S-SAS). Hence $\hat{1} \cong \hat{2}$. Since $\hat{1}$ and $\hat{2}$ are (corresponding left figure, and alternate right figure) congruent, l is parallel to l'.

Theorem 5.1.3

The product of homothecies $\mathcal{D}_{O,\lambda}$ and $\mathcal{D}_{O,\mu}$ is a homothecy $\mathcal{D}_{O,\lambda\mu}$. Consequently, $\mathcal{D}_{O,\lambda}^{-1} = \mathcal{D}_{O,\frac{1}{\lambda}}$.

Proof:

Clearly, $\mathcal{D}_{O,\lambda\mu}(O) = \mathcal{D}_{O,\lambda}(O) = \mathcal{D}_{O,\mu}(O) = O$. If P is any other point, then $\mathcal{D}_{O,\lambda}(P) = P'$ with $\left|\overline{OP'}\right| = |\lambda|\,|\overline{OP}|$ with O, P, P' collinear. Aslo, $\mathcal{D}_{O,\mu}(P') = P''$ with $\left|\overline{OP''}\right| = |\mu|\left|\overline{OP'}\right|$ with O, P', P'' collinear. Hence O, P, P'' are collinear and

$$\left|\overline{OP''}\right| = |\mu|\left|\overline{OP'}\right| = |\mu|\,(|\lambda|\,|\overline{OP}|) = |\lambda\mu|\;|\overline{OP}|.$$

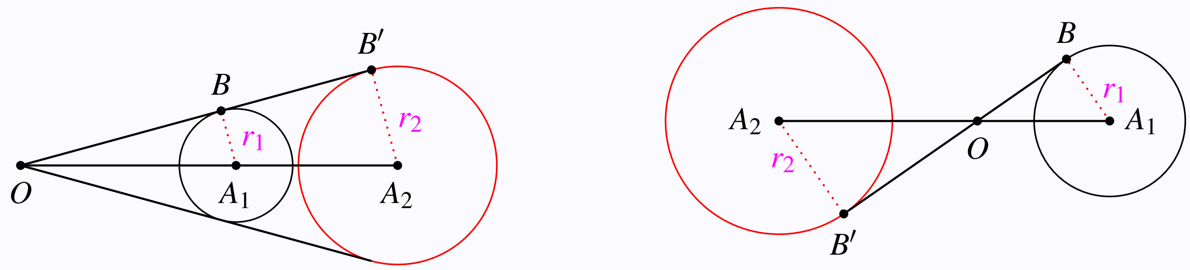
That is $\mathcal{D}_{O,\lambda\mu}(P) = P''$.

Therefore, $\mathcal{D}_{O,\lambda}\mathcal{D}_{O,\frac{1}{\lambda}} = \mathcal{D}_{O,\lambda\frac{1}{\lambda}} = \mathcal{D}_{O,1} = \mathbf{I}$.

Example 5.1.3

Let $\odot A_1$ and $\odot A_2$ be two circles with two distinct centers $A_1 \neq A_2$ with two different radii $r_1 \neq r_2$. Show that there is exactly two homothecies $\mathcal{D}_{O_1,\lambda_1}, \mathcal{D}_{O_2,\lambda_2}$ that map $\odot A_1$ to $\odot A_2$. Construct the centers O_1 and O_2 of such homothecies.

Solution:

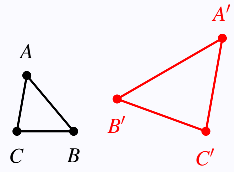


Note that the homothecy ratio is positive (direct) or negative (opposite). Then there is only two cases as drawn above. The homothecy center would be the intersection point of $\overline{A_1A_2}$ and $\overline{BB'}$, and the ratio is then $|\lambda| = \frac{r_2}{r_1}$ assuming that $\mathcal{D}_{O,|\lambda|}(B) = B'$ and $\mathcal{D}_{O,|\lambda|}(A_1) = A_2$. Hence $\mathcal{D}_{O,|\lambda|}$ maps $\odot A_1$ to $\odot A_2$ in both cases.

5.2 Similarity

Definition 5.2.1

Let k be a positive scalar. A **similarity** with ratio k is the transformation $\mathcal{S}_k$ such that for any points A and B with $A' = \mathcal{S}_k(A)$ and $B' = \mathcal{S}_k(B)$, $\left|\overline{A'B'}\right| = k\left|\overline{AB}\right|$.



Remark 5.2.1

- A similarity has no center.
- Every isometry is a similarity of ratio 1.
- Every homothecy $\mathcal{D}_{O,\lambda}$ is a similarity of ratio $|\lambda|$.
- The product of two similarities of ratios k_1, k_2 is a similarity of ratio k_1k_2. See Theorem 5.1.3.
- The inverse of $\mathcal{S}_k$ is $\mathcal{S}_{\frac{1}{k}}$.

Another definition of a similarity:

Definition 5.2.2

A **similarity** is a composition of a finite number of dilations or isometries. The **ratio** of a similarity is the product of the ratios of the dilations in the composition. If there are no dilations in the composition, the ratio is defined to be 1.

Two figures in a plane are **similar** if there exists a similarity transformation taking one figure onto the other figure.

Remark 5.2.2

Some examples of similarities:

- A **dilative reflection** is a similarity produced by a dilation (homothecy) and a reflection.
- A **dilative rotation** is a similarity produced by a dilation (homothecy) and a rotation.
- A **dilative translation** is a similarity produced by a dilation (homothecy) and a translation.

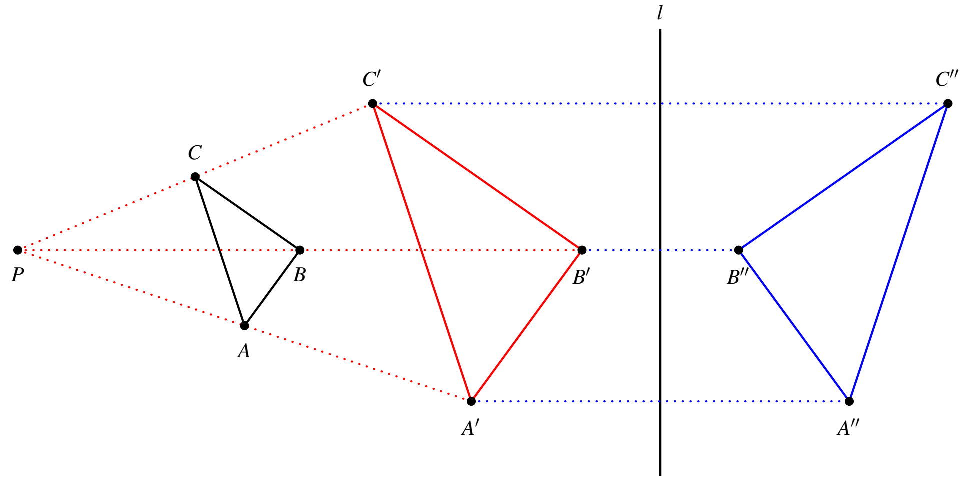


Figure 5.2: Dilative reflection: $\mathbf{R}_l \circ \mathcal{D}_{P,\lambda}\,(\triangle\, ABC)$.

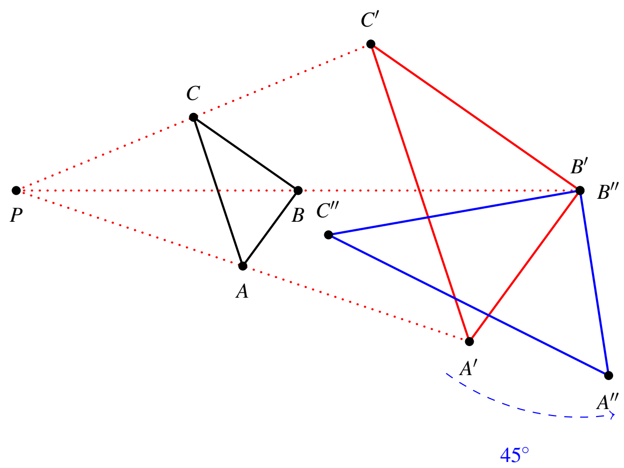


Figure 5.3: Dilative rotation: $\mathcal{R}_{B',45^\circ} \circ \mathcal{D}_{P,\lambda}\,(\triangle\, ABC)$.

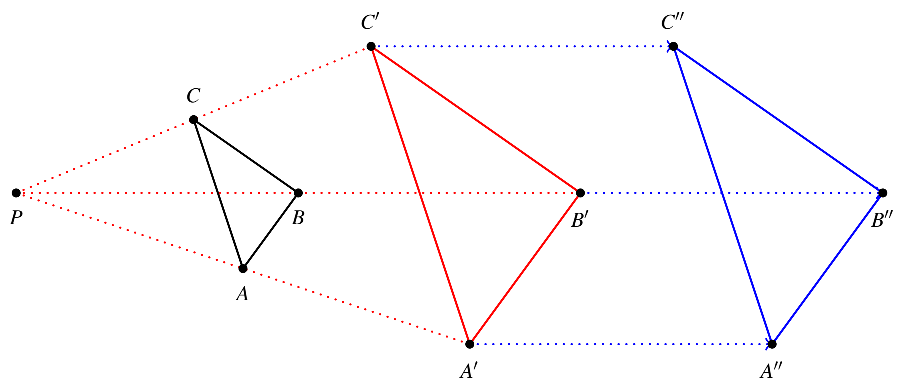


Figure 5.4: Dilative translation: $\mathcal{T}_{\overrightarrow{B'B''}} \circ \mathcal{D}_{P,\lambda}\,(\triangle\, ABC)$.

Example 5.2.1

Let $\mathbf{T}$ be a transformation of the plane. Show that if $\mathbf{T}$ preserves angle measure, then $\mathbf{T}$ is a similarity.

Solution:

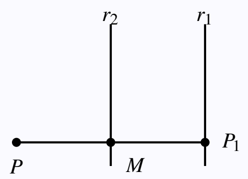
Let $\triangle ABC$ be a triangle with $\mathbf{T}(\triangle ABC) = A'B'C'$. Then $\hat{A} \cong \hat{A}', \hat{B} \cong \hat{B}', \hat{C} \cong \hat{C}'$ and hence $\triangle ABC \sim \triangle A'B'C'$. That is,

$$\frac{\left|\overline{A'B'}\right|}{\left|\overline{AB}\right|} = \frac{\left|\overline{A'C'}\right|}{\left|\overline{AC}\right|} = \frac{\left|\overline{B'C'}\right|}{\left|\overline{BC}\right|} = k.$$

Therefore, $\mathcal{S}_k(\triangle ABC) = \triangle A'B'C'$ and it is a similarity of ratio k.

Example 5.2.2

Let $\mathcal{S}_\lambda$ be a similarity and P any point in the plane. Show that there exists a translation $\mathbf{T}$ such that P is invariant under $\mathbf{T}\mathcal{S}_\lambda$.



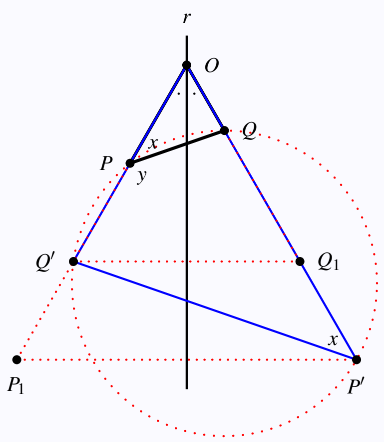
Solution:

If $\lambda = 1$, then simply $\mathcal{S}_1(P) = P$. Take $\mathbf{T} = \mathbf{I}$ to get $\mathbf{T}\mathcal{S}_\lambda(P) = P$.

If $\lambda \neq 1$, then $\mathcal{S}_\lambda(P) = P_1 \neq P$. Let M be the midpoint of $\left|\overline{PP_1}\right|$ and let $\mathbf{T} = \mathbf{R}_{r_2}\mathbf{R}_{r_1}$ where r_2 and r_1 passing through M and P_1 so that $\overline{PP_1}$ is perpendicular to both lines. That is $r_1 \parallel r_2$. Note that $\mathbf{T}(P_1) = \mathbf{R}_{r_2}\mathbf{R}_{r_1}(P_1) = \mathbf{R}_{r_2}(P_1) = P$. Therefore, $\mathbf{T}(\mathcal{S}_\lambda(P)) = \mathbf{T}(P_1) = P$.

Example 5.2.3

Let r be the angle bisector of angle O of a triangle $\triangle POQ$. Consider the dilative reflection $\mathbf{T} = \mathbf{R}_r \circ \mathcal{D}_{O,\lambda}$. If $\mathbf{T}(P) = P'$ and $\mathbf{T}(Q) = Q'$, show that the quadrilateral $PQP'Q'$ is cyclic.



Solution:

Let $\mathcal{D}_{O,\lambda}(P) = P_1$ and $\mathbf{R}_r(P_1) = P'$; and let $\mathcal{D}_{O,\lambda}(Q) = Q_1$ and $\mathbf{R}_r(Q_1) = Q'$. That is $\mathbf{T}(P) = P'$ and $\mathbf{T}(Q) = Q'$. Note that $x + y = 180°$. Recall that reflection and homothecy preserve angle measure, and hence $\left|O\hat{P}Q\right| = \left|O\hat{P_1}Q_1\right| = \left|O\hat{P'}Q'\right| = x$. Therefore, $\left|Q'\hat{P'}Q\right| + \left|Q\hat{P}Q'\right| = 180$. Similarily, we can show that $\left|P\hat{Q}P'\right| + \left|P'\hat{Q'}P\right| = 180$. Thus, $PQP'Q'$ has supplementary opposite angles and hence it is cyclic.

Chapter

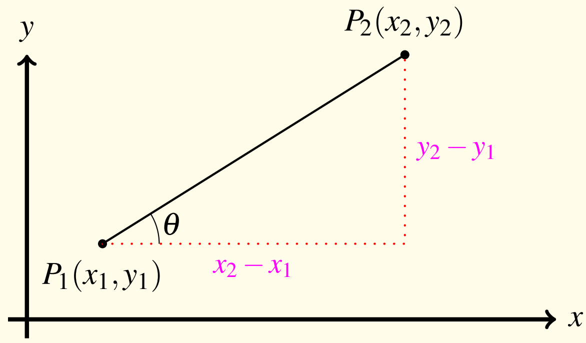
6 Coordinate Geometry

6.1 Coordinates of Points and Lines

Remark 6.1.1

- A point $A(x,y)$ in the **Cartesian** plane (or xy-plane) is represented by its x and y coordinates.
- The **slope** of a line l, denoted m_l, through the points $P_1(x_1,y_1)$ and $P_2(x_2,y_2)$ is defined by

$$m_l = \frac{y_2-y_1}{x_2-x_1} = \frac{\text{changes in } y}{\text{changes in } x} = \tan\theta.$$



Remark 6.1.2

A line l can be presented by:

1. **standard form**: $ax+by+c=0$, where a and b are not both zeros.
2. **slope-intercept form**: $y=mx+c$, where m is the slope of the line and c is y-intercept.
3. **point-slope form**: $(y-y_1)=m(x-x_1)$, where (x_1,y_1) is a point on the line l with slope m.

Theorem 6.1.1

Let l_1 and l_2 be two lines with slopes m_1 and m_2, respectively. Then:

1. $l_1 \parallel l_2$ if and only if $m_1 = m_2$.
2. $l_1 \perp l_2$ if and only if $m_1 \cdot m_2 = -1$ if and only if $m_2 = -\frac{1}{m_1}$.

Example 6.1.1

Find the slope of the line l passing through points $A(2,-3)$ and $B(1,5)$ and write its equation.

Solution:

Simply $m_l = \frac{5-(-3)}{1-2} = \frac{8}{-1} = -8$.

Hence $l:\ (y-5) = -8(x-1)$ or $l:\ y = -8x+13$ or $l:\ 8x+y-13=0$.

Example 6.1.2

Let $l_1:\ 2x+y=1$; $l_2:\ 2y-x=7$; $l_3:\ 4x+2y=0$; $l_4:\ y=2$; $l_5:\ y=7$; $l_6:\ x=-2$; and $l_7:\ x=2$.

Then: $m_1 = \frac{-2}{1} = -2$; $m_2 = \frac{1}{2}$; $m_3 = -2$; $m_4 = m_5 = 0$; $m_6 = m_7 =$ undefined.

Therefore: $l_1 \parallel l_3$ and $l_1 \perp l_2 \perp l_3$. Also, $l_4 \parallel l_5$ (horizontal lines) and $l_6 \parallel l_7$ (vertical lines).

Hence l_4 and l_5 are perpendicular to l_6 and l_7.

Definition 6.1.1

Let $A(x_1,y_1)$ and $B(x_2,y_2)$ be two points and let $l:\ ax+by+c=0$ be a line. Then

- The **distance** between A and B is defined by

$$d(A,B) = \sqrt{(x_1-x_2)^2+(y_1-y_2)^2}.$$

- The **distance** between A and line l is defined by

$$d(A,l) = \frac{|ax_1+by_1+c|}{\sqrt{a^2+b^2}}.$$

- The **midpoint** of the segment $\overline{AB}$ is defined by

$$\text{mid } \overline{AB} = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right).$$

Example 6.1.3

Find $d(A,l)$, where $A(1,2)$ and $l:\ y=2x-1$.

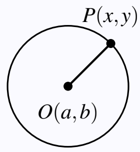
Solution:

Clearly, $l:\ 2x-y-1=0$ and hence $d(A,l) = \frac{|2(1)-(2)-1|}{\sqrt{2^2+1^2}} = \frac{1}{\sqrt{5}}$.

Definition 6.1.2

The **equation of the circle** with center $A(a,b)$ and radius r is

$$(x-a)^2+(y-b)^2=r^2.$$



Example 6.1.4

Find the locus of points equidistant from $A(3,-2)$ and $B(4,3)$.

Solution (1):

Let $M(x,y)$ be the points of the locus. Thus, $d(M,A)=d(M,B)$. That is

$$\begin{gathered}
\sqrt{(x-3)^2+(y+2)^2}=\sqrt{(x-4)^2+(y-3)^2}\\
(x-3)^2+(y+2)^2=(x-4)^2+(y-3)^2\\
\left(x^2-6x+9\right)+\left(y^2+4y+4\right)=\left(x^2-8x+16\right)+\left(y^2-6y+9\right)\\
2x+10y-12=0.
\end{gathered}$$

Thus, the locus of points equidistant from A and B are the points of the line $l:\ 2x+10y-12=0$.

Solution (2):

We can solve the question in a different way: Recall that the locus of points M equidistant from two points is a line l which is the perpendicular bisector of $\overline{AB}$. Clearly the slope of $\overleftrightarrow{AB}$ is $m_{\overline{AB}}=5$ and hence $m_l=-\frac{1}{5}$. Also, $M=\text{mid}\ \overline{AB}$ lies on the line l where $M=\left(\frac{7}{2},\frac{1}{2}\right)$. Therefore, the locus is the equation of

$$l:\left(y-\frac{1}{2}\right)=-\frac{1}{5}\left(x-\frac{7}{2}\right)\ \Rightarrow\ 10y-5=-2x+7\ \Rightarrow\ 2x+10y-12=0.$$

Example 6.1.5

Find the locus of points $P(x,y)$ that are at distance 3 cm from the point $A(1,2)$.

Solution:

The locus of points $P(x,y)$ at distance 3 cm is the points of the circle centered at A with radius 3 cm. That is, $3=d(P,A)=\sqrt{(x-1)^2+(y-2)^2}$. Hence, the locus is the circle with equation: $(x-1)^2+(y-2)^2=9$.

Example 6.1.6

Find the locus of points M equidistant from the lines $l_1 : \; x-y+1=0$ and $l_2 : \; 2x-2y+7=0$.

Solution:

Notice that if $l_1 \parallel l_2$, then the locus is a line that is parallel to both lines l_1 and l_2. Otherwise, the locus is two lines which are angle bisectors of the two lines.

Let $M(x,y)$ be the points of the locus. Thus,

$$d(M,l_1)=d(M,l_2) \;\Rightarrow\; \frac{|x-y+1|}{\sqrt{1+1}}=\frac{|2x-2y+7|}{\sqrt{4+4}} \;\Rightarrow 2\sqrt{2}|x-y+1|=\sqrt{2}|2x-2y+7|.$$

That is we have two cases:

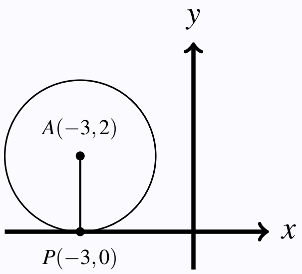
Case 1: $2(x-y+1)=+(2x-2y+7)$, and hence $2=7$ which is impossible. So this case is rejected.

Case 2: $2(x-y+1)=-(2x-2y+7)$, and hence $4x-4y+9=0$.

Therefore, the locus is formed by the line with equation: $4x-4y+9=0$. We obtain here that l_1 and l_2 are parallel.

Example 6.1.7

Find an equation of the circle with center $A(-3,2)$ and tangent to the x-axis.



Solution:

Since the circle is tangent to x-axis, we have the radius equals to the y-coordinates of A which is the distance from x-axis to A. Thus, $r=|2|=2$, and hence the circle equation: $(x+3)^2+(y-2)^2=4$.

6.2 Transformation in Coordinates Geometry

Remark 6.2.1

If $P(a,b)$ is a point, then its reflection in a line is:

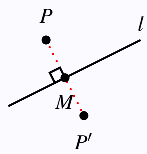
- **reflection in x-axis**: $P(a,b) \mapsto P'(a,-b)$.
- **reflection in y-axis**: $P(a,b) \mapsto P'(-a,b)$.
- **reflection in the origin**: $P(a,b) \mapsto P'(-a,-b)$.
- **reflection in the line $y=x$**: $P(a,b) \mapsto P'(b,a)$.
- **reflection in the line $y=-x$**: $P(a,b) \mapsto P'(-b,-a)$.

The reflection of a point $P(a,b)$ in a general line $y = mx + c$ can be computed using the definition of reflection. See Example 6.2.1.

Example 6.2.1

Find $\mathbf{R}_l(P)$, where $l:\ x-2y+3=0$ and $P(2,5)$.

Solution:



Let $P'(x_1,y_1) = \mathbf{R}_l(P)$. The midpoint $M = \text{mid}\,\overline{PP'} = \left(\frac{x_1+2}{2}, \frac{y_1+5}{2}\right) \in l$ by the definition of reflection. That is, $\frac{x_1+2}{2} - 2\frac{y_1+5}{2} + 3 = 0$. Multiply both sides by 2 to get $x_1+2-2y_1-10+6=0$ and hence $x_1-2y_1-2=0 \cdots (1)$.
Since, $\overline{PP'} \perp l$ and $m_l = \frac{1}{2}$, we get the slope of $\overline{PP'} = -2$. Thus $\overline{PP'}:\ (y_1-5) = -2(x_1-2)$ and hence the equation of the line $\overline{PP'}:\ 2x_1+y_1-9=0 \cdots (2)$.
Computing $(1)+2\cdot(2)$, we obtain $5x_1-20=0$ and hence $x_1=4$ and thus $y_1=1$. That is $P'(4,1)$.

Example 6.2.2

If $P'(4,6)$ is the image of $P(0,2)$ under $\mathbf{R}_l$, then find an equation of the line l.

Solution:

Note that the slope of $\overline{PP'}$ is $\frac{6-2}{4-0} = 1$ and hence $m_l = -1$. Moreover, $M = \text{mid}\,\overline{PP'} = \left(\frac{4+0}{2}, \frac{6+2}{2}\right) = (2,4) \in l$. Thus: $l:\ (y-2) = -(x-4)$ and hence $l:\ y+x-6=0$.

Remark 6.2.2

The translation of the point $P(x,y)$ of a horizontal units and b vertical units is $P'(x+a,y+b)$. That is,

$$\mathcal{T}_{a,b} : P(x,y) \mapsto P'(x+a,y+b).$$

Example 6.2.3

Show that the product of two translations is a translation.

Solution:

Let $\mathcal{T}_{a_1,b_1}$ and $\mathcal{T}_{a_2,b_2}$ be two translations. Then we show that $\mathbf{T} = \mathcal{T}_{a_1,b_1} \circ \mathcal{T}_{a_2,b_2}$ is a translation. For any point (x,y), we have

$$\begin{aligned}\mathbf{T}(x,y) &= \mathcal{T}_{a_1,b_1}\left(\mathcal{T}_{a_2,b_2}(x,y)\right)\\ &= \mathcal{T}_{a_1,b_1}(x+a_2,y+b_2) = (x+a_2+a_1,y+b_2+b_1)\\ &= \mathcal{T}_{a,b}(x,y),\end{aligned}$$

where $a = a_1+a_2$ and $b = b_1+b_2$ which is also a translation.

Remark 6.2.3

The rotation of $P(x,y)$ about the origin through angle θ is $P'(x',y')$, where

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

That is, $x' = x\cos\theta - y\sin\theta$ and $y' = x\sin\theta + y\cos\theta$ and the matrix $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ is called the **rotation matrix**. Observe that:

- $\mathcal{R}_{O,\frac{\pi}{2}}((x,y)) = (-y,x)$.
- $\mathcal{R}_{O,\pi}((x,y)) = (-x,-y)$.

Note that a half-turn is the same as reflecting in origin.

Example 6.2.4

Find the rotation of $P(1,5)$ about the origin through $\frac{\pi}{6}$.

Solution:

Using the rotation matrix, we get:

$$\mathcal{R}_{O,\frac{\pi}{6}}((1,5)) = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 5 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} - \frac{5}{2} \\ \frac{1}{2} + \frac{5\sqrt{3}}{2} \end{bmatrix}.$$

Example 6.2.5

If a rotation $\mathcal{R}_{(0,0),x}$ maps $A(3,-4)$ to $A'(4,3)$, then find the measure of x.

Solution:

Using the rotation matrix D, we have $A' = D\,A$. That is:

$$\begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{bmatrix} \begin{bmatrix} 3 \\ -4 \end{bmatrix}.$$

Hence,

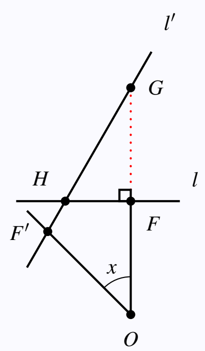
$$4 = 3\cos x + 4\sin x \cdots (1) \quad \text{and} \quad 3 = 3\sin x - 4\cos x \cdots (2).$$

Computing $4\cdot(1) + 3\cdot(2)$, we get $25\sin x = 25$ and hence $\sin x = 1$. Therefore, $x = \frac{\pi}{2}$.

Example 6.2.6

The rotation $\mathcal{R}_{O,x}$ maps the line l to line l'. Show that one of the angles between l and l' has measure x.

Solution:



Let F be a point on l so that $\overline{OF} \perp l$. Thus, $\mathcal{R}_{O,x}(F) = F' \in l'$.

Let G be the point of intersection of $\overline{OF}$ with l', and let H be the intersection point of l with l'.

Since $\left|O\hat{F}H\right| = 90^\circ$, we have $\left|O\hat{F'}H\right| = 90^\circ$ (since rotation preserves angle measure). Thus we get $\left|F'\hat{G}O\right| = 90^\circ - x$.

Therefore, $\left|G\hat{H}F\right| = 90^\circ - (90^\circ - x) = x$.

Example 6.2.7

Find the image of $P(2,3)$ under $\mathcal{H}_{(-2,7)}$. [Or: Find $\mathcal{R}_{(-2,7),\pi}((2,3))$]

Solution:

Let $\mathcal{H}_{(-2,7)}(P) = P'(x,y)$. Then, the midpoint $M = \text{mid } \overline{PP'} = \left(\frac{2+x}{2}, \frac{3+y}{2}\right) = (-2,7)$. That is, $-2 = \frac{2+x}{2}$ and $7 = \frac{3+y}{2}$ which implies that $x = -6$ and $y = 11$. Hence $P'(-6,11)$.

Remark 6.2.4

The homethecy (dilation) image of point $P(x,y)$ with center O and ratio λ is $P'(\lambda x, \lambda y)$. That is,

$$\mathcal{D}_{O,\lambda} : P(x,y) \mapsto P'(\lambda x, \lambda y).$$

Example 6.2.8

If $\mathcal{D}_{O,\lambda}$ maps $\overline{PQ}$ to $\overline{P'Q'}$, show that $\overline{PQ} \parallel \overline{P'Q'}$.

Solution:

We show that the slopes of $\overline{PQ}$ and $\overline{P'Q'}$ are equal. Note that $\mathcal{D}_{O,\lambda}(P(x_1,y_1)) = P'(\lambda x_1, \lambda y_1)$ and $\mathcal{D}_{O,\lambda}(Q(x_2,y_2)) = Q'(\lambda x_2, \lambda y_2)$. Hence, the slope $\overline{P'Q'}$ is

$$\frac{\lambda y_2 - \lambda y_1}{\lambda x_2 - \lambda x_1} = \frac{\lambda(y_2 - y_1)}{\lambda(x_2 - x_1)} = \frac{y_2 - y_1}{x_2 - x_1} = \text{ slope of } \overline{PQ}.$$

Thus, $\overline{PQ} \parallel \overline{P'Q'}$.

The Index

O

P

Q

R

S

T